APPLIED BESSEL FUNCTIONS

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By the Same Author

APPLIED DIFFERENTIAL EQUATIONS

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PREFACE

This little book, like so many of its kind, had its roots in a course of lectures. I gave the lectures, by request, to an audience consisting mainly of graduates in physics, soil mechanics and the various branches of civil, mechanical and electrical engineering. The listeners had begun to find that progress in their researches and further studies was being impeded by their inability to handle Bessel functions; it was my task to remedy this defect in their equipment as best I could.

Somewhat to my surprise the audience turned up in considerable strength; to my even greater surprise a high percentage stayed the course without hope of ulterior reward beyond the acquirement of wisdom. I was flattered into believing with venial weakness that I might have done worse; and maybe I lent all too readily "a credent ear" to the suggestion that the lectures be put into book form. However, now that the step has been taken, I am under a moral obligation to answer two questions.

The book is definitely mathematical and is addressed to technicians. This immediately poses the question, what should be the mathematical equipment of the man who hopes to benefit by reading it? The lectures were designed to fall within the circle of ideas of those who had taken the mathematical course that usually goes with a degree course in physics or engineering. More specifically, no use was to be made of contour integration or of the complex variable in the analytic sense. On the other hand, I assumed a knowledge of ordinary differential equations with constant coefficients, such as occur in the theory of beams or of simple circuits. I further presumed my audience to know that such partial differential equations as normally arise can be solved as the product of functions of the independent variables and might involve Fourier series.

I embarked on the lectures with a good deal of trepidation, for reasons which will be patent to anyone who has ever essayed the task. The main difficulty is to establish the existence and nature of the zeros of the Bessel functions. The series representation of a function is often the least informative thing about it, except for purposes of computation, and I had little hesitation in relegating that to a subordinate place in the scheme of exposition. The recurrence formulæ for any function containing a parameter are certain to play an important part in its applications; but the recurrence formulæ are most likely to be derived from an integral representation of the function, and that is something which definitely has to be considered advanced, even in the study of differential equations.

This brings me to the second of the two questions which I mentioned previously. It is to account for the unusual lay-out of the course. After much cogitation I decided that the special circumstances of the task justified the heterodoxy of making the recurrence formulæ the starting point. I had no reason to regret my decision, for as pointed out in the bibliographical note, I found I had the moral support of the argute E. B. Wilson.

I inserted the initial chapter, treating the ancillary functions, on pedagogical grounds. It is better to do a little preliminary spadework to make sure that the soil is ready for sowing, rather than to interrupt one's discourse by parenthetic paragraphs and distracting footnotes. There is the additional defence that the Gamma function is in itself sufficiently interesting to justify the expenditure of a few hours on mastering its salient properties.

Coming to the main obstacle, how to demonstrate the existence and nature of the zeros of the Bessel functions, I naturally inquired how other writers had approached it. In three current books I found that one employed Bessel's original method that a beginner would hardly relish; the second advocated the uninspired method of plotting from tabulated values; the third went on the principle *ignotum per ignotius* by borrowing the unproven asymptotic values. That removed all qualms I might have had about using the oscillation theory. The majority of English mathematicians are indebted to the standard treatise by G. N. Watson rather than to the continental school, and I think that if its author had felt more drawn to the Sturmian theory, other writers would have adopted it. In any case, mathematical physics has in recent years given the theory a renascence, and my audience was pleased to learn that a differential equation can be made to impart information by methods other than solving it. The slight advance on what is usually taught concerning differential equations is given in the second chapter, and the unbiased can hardly fail to recognize in it an instrument admirably suited to its purpose.

The year of publication marks a centenary; F. W. Bessel died at Koenigsberg on 17th March, 1846.

Concerning tables, it is confidently to be expected that the British Association Tables Committee will shortly issue a second volume on Bessel functions in continuation of their previous work. The new volume will tabulate the four functions J, Y, I and K, the argument running from 0 to 25 and the integral order from 2 to 20. The entries will each contain 8 reliable figures and there will be provision for interpolation.

WATFORD,
July, 1946.

F. E. RELTON.

CONTENTS

The paragraphs are numbered consecutively throughout each chapter; the equations are numbered consecutively throughout each paragraph. Thus the reference 7·3(4) means chapter 7, paragraph 3, equation (4) therein.

CHAPTER I

The Error Function; Beta and Gamma Functions

1·1. The study of functions.

The functions that one encounters early in one's studies are usually termed "the elementary functions". They include the trigonometrical functions, the hyperbolic functions, the exponential and logarithmic function. Functions other than these are termed "transcendental functions" and for the most part they fall into one or other of two categories; either they are defined by an integral or they satisfy a differential equation. It occasionally happens that a function falls into both categories.

A mathematician, faced with the study of a new function, has very much the same attitude that a physicist or a chemist would adopt towards a substance presented for examination. The latter would inquire whether the substance was crystalline or necessarily amorphous; whether it was soluble and what were its solvents; what was its density and thermal conductivity, and so on. A mathematician would inquire whether there were any values of the variable for which the function vanished, or alternatively became infinite; in his own terminology, he would seek for zeros and poles. He would be interested to know whether the function had an addition formula, by which we mean an expression for $f(a + b)$ in terms of $f(a)$ and $f(b)$, the simpler the better, and preferably algebraic. Thus the trigonometrical function $\sin x$ has the algebraic addition formula

$$\sin(x + y) = \sin x(1 - \sin^2 y)^{\frac{1}{2}} + \sin y(1 - \sin^2 x)^{\frac{1}{2}},$$

which can also be written in the differential form

$$\sin(x + y) = \sin x \frac{d}{dy} \sin y + \sin y \frac{d}{dx} \sin x.$$

The mathematician would be interested to know whether the function had maxima and minima, and in particular whether it was asymptotic to some simple expression for large values of the variable, just as $\cosh x$ is asymptotic to $\frac{1}{2}e^x$. Finally he may decide to tabulate the function, though this is usually the last thing he does and many functions have been studied which nobody has yet troubled to tabulate.

The investigation of these and other points of interest is rarely carried out by evaluating the aforementioned integral or by solving the differential equation which defines the function. There are other methods of approach and we must be prepared to encounter these in the following pages. We shall be concerned almost solely with the so-called Bessel functions, of which there are several kinds; and it may as well be stated at the outset that the graph of a Bessel function usually looks like the distorted sine-curve that represents a damped oscillation. The amplitude steadily decreases and there is this difference, that the "period" is not quite constant, so that the function cannot legitimately be termed periodic. But before we come to the Bessel functions there is a certain amount of spadework to be done.

1·2. A useful limit.

We begin with a limit for which we shall find almost immediate use. If n is a positive number we have

(1) $$x^n e^{-x} \to 0, \ x \to 0, \ n > 0.$$

The arrow notation signifies "tends to", "approaches", or "has the limit". The above line signifies that when n is positive and x tends to zero, then the expression at the left also tends to zero. The result is obviously correct since e^{-x} tends to unity and x^n tends to zero. Similarly we have the not quite so obvious result

(2) $$x^n e^{-x} \to 0, \ x \to \infty, \ n > 0.$$

Here x^n tends to become indefinitely large and e^{-x} tends to zero. We establish the truth of the statement by writing

(3) $$x^n e^{-x} = \frac{x^n}{e^x} = \frac{x^n}{1 + x + x^2/2 + \ldots}.$$

On dividing numerator and denominator by x^n the numerator becomes unity. The denominator becomes a limited succession of negative powers followed by an infinite succession of positive powers of x. The negative powers tend to zero and the positive powers tend to become infinite. Thus the fraction tends to zero and the truth of (2) is established.

EXERCISES

1. Sketch the graph of $x^n e^{-x}$ when x is positive and n is (i) positive, (ii) zero, (iii) negative. Prove that in (i) there is a single maximum; but in (ii) and (iii) the value steadily decreases as x increases, whilst in (iii) the initial value is infinite.

2. Prove that $\dfrac{\log x}{x} \to \infty, x \to 0$. Also $\dfrac{\log x}{x} \to 0, x \to \infty$.

3. Deduce that as the number n tends to become very large, the nth root of n tends to unity.

4. Prove that the series $\dfrac{\log 1}{1} + \dfrac{\log 2}{2} + \dfrac{\log 3}{3} + \dots$ is divergent and deduce that the product $1^1 . 2^{\frac{1}{2}} . 3^{\frac{1}{3}} . 4^{\frac{1}{4}} \dots$ is infinite.

5. Prove that as $a \to \infty$, $a^2 e^{-a^2} \to 0$.

6. Sketch roughly the graph of the function $(\log x)/x$ for positive values of x. Prove that it has a single maximum value e^{-1}.

1·3. The error function, Erf(x).

The error function is one of the class of functions defined by an integral. It plays a prominent part in the theory of probability, refraction of light, conduction of heat, and so on. Its definition is

$$y = \operatorname{Erf}(x) = \int_0^x e^{-t^2}\,dt. \qquad (1)$$

The graph of the integrand is shown in fig. 1. It is symmetrical and has a maximum value of unity when t is zero; it is doubly asymptotic to the horizontal axis and has two inflections. The variable t is a dummy and could equally well be called θ or any other symbol. The integral represents the area included between ordinates at the origin and at some variable distance x. Its value will be found tabulated in almost

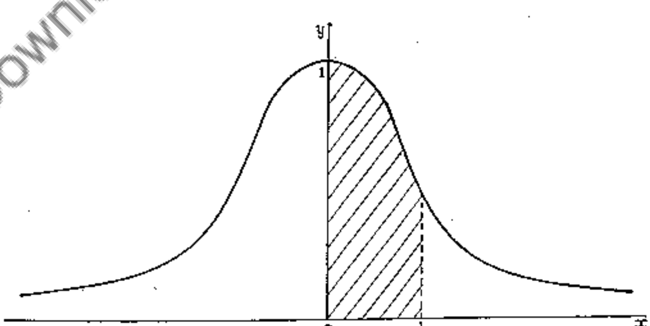

Fig. 1.—Graph of function $y = e^{-x^2}$

any work on probability and in many sets of tables. The integral can be evaluated by expanding the exponential in an infinite series of ascending powers of t and integrating term by term; but the result is not very informative.

It not infrequently happens that where an indefinite integral cannot be evaluated in a succinct form, a neat result can be obtained when the integral is definite. We now propose to show that the whole area under the curve in the first quadrant is finite and has the value

$$\int_0^\infty e^{-t^2}dt = \tfrac{1}{2}\sqrt{\pi}. \tag{2}$$

This last integral is called an infinite integral on account of the upper limit; if it exists it is regarded as the limit of the same integral, taken between 0 and a, when a is made indefinitely large. Hence we may write

$$N_a = \int_0^a e^{-x^2}dx = \int_0^a e^{-y^2}dy,$$

$$N = \int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-y^2}dy;$$

$$N_a \to N \text{ as } a \to \infty.$$

Here N is a pure number and N_a is a number dependent on a. We may write

$$N_a{}^2 = \int_0^a e^{-x^2}dx \int_0^a e^{-y^2}dy = \int_0^a \int_0^a e^{-x^2-y^2}dx\,dy,$$

the double integral being taken over a square of side a. On converting to polar co-ordinates r, θ we have $e^{-x^2-y^2} = e^{-r^2}$ and the area-element $dx\,dy$ is replaced by $r\,dr\,d\theta$. The limits for r are 0 and a, and the limits for θ are 0 and $\frac{1}{2}\pi$; but as the polar co-ordinates cover only a quadrant instead of a square we write

$$N_a{}^2 = \int_0^{\frac{1}{2}\pi}\int_0^a e^{-r^2}r\,dr\,d\theta + R,$$

where R is known as the remainder. It represents an addition due to the area between the quadrant and the square (fig. 2), and our first step is to show that the value of R is ultimately negligible, or $R \to 0$ as $a \to \infty$. Here is the proof. At any point within this area we have

$$x^2 + y^2 > a^2,\ e^{-x^2-y^2} < e^{-a^2},$$

and since the area over which this is to be integrated is certainly less than half the square, we have $R < \frac{1}{2}a^2e^{-a^2}$. Hence $R \to 0$ as $a \to \infty$. We may now let a become infinite and rewrite a previous line as

$$N^2 = \int_0^{\frac{1}{2}\pi} d\theta \int_0^{\infty} re^{-r^2} dr = \tfrac{1}{2}\pi\left[-\tfrac{1}{2}e^{-r^2}\right]_0^{\infty} = \tfrac{1}{4}\pi,$$

whence $N = \frac{1}{2}\sqrt{\pi}$ as stated earlier.

As an example of its application, consider the following intractable-looking integral which occurs in the theory of heat conduction.

$$I = \int_0^{\infty} e^{-x^2-a^2/x^2} dx, \text{ so that } \frac{dI}{da} = -2a\int_0^{\infty} e^{-x^2-a^2/x^2}\frac{dx}{x^2}.$$

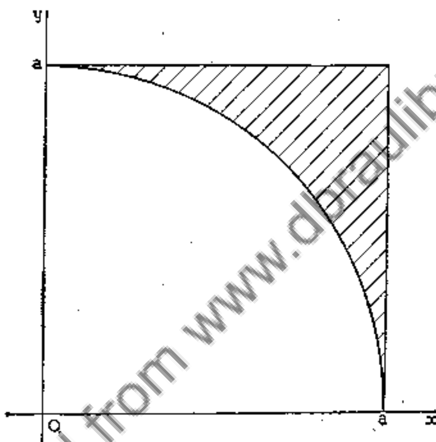

Fig. 2

The substitution $xy = a$ is equivalent to

$$\frac{dx}{x^2} = -\frac{dy}{a}$$

whence

$$\frac{dI}{da} = 2a\int_{\infty}^{0} e^{-y^2-a^2/y^2}\frac{dy}{a} = -2I.$$

The solution of this simple differential equation is $I = ce^{-2a}$. The arbitrary constant c can be determined by putting $a = 0$. This gives $c = \frac{1}{2}\sqrt{\pi}$ and $I = \frac{1}{2}\sqrt{\pi}e^{-2a}$.

EXERCISES

1. Sketch roughly the graph of $y = \operatorname{Erf}(x)$ for positive values of x. It closely resembles the right half of $y = \tanh x$; at what angle does it leave the origin?

2. If $I = \int_0^\infty e^{-x^2} \cos 2bx\, dx$, verify that $\frac{dI}{db} + 2bI = 0$; hence deduce that $I = \frac{1}{2}\sqrt{\pi}e^{-b^2}$.

3. By a slight change of variable, deduce that $\int_0^\infty e^{-a^2x^2} \cos 2bx\, dx = \frac{\sqrt{\pi}}{2a} e^{-b^2/a^2}$.

4. Integrating No. 2 between 0 and b, deduce that

$$\int_0^\infty e^{-x^2} \frac{\sin 2bx}{x}\, dx = \sqrt{\pi}\left\{b - \frac{b^3}{3} + \frac{b^5}{5.2!} - \ldots\right\}.$$

5. Use the substitution $x = t - a$ to establish the result

$$\int_a^\infty e^{2at-t^2} dt = \tfrac{1}{2}\sqrt{\pi}e^{a^2}.$$

6. The error function is sometimes defined as $\operatorname{Erfc}(x) = \int_x^\infty e^{-t^2} dt$. Deduce $\operatorname{Erf}(x) = \frac{1}{2}\sqrt{\pi} - \operatorname{Erfc}(x)$.

(The function is sometimes called the "normal error" or "probability integral" and is tabulated in the form $I = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt$. This is asymptotic to unity for large values of x and reaches the value 0·99998 when $x = 3$.)

1·4. The Gamma function.

The Gamma function is one of the class defined by an integral. It is known that it does not satisfy any differential equation with rational coefficients and it is sometimes called the Eulerian integral of the second kind. It is one of the earliest transcendental functions that one encounters and it is certainly one of the most interesting.

If n is a positive number, we have as the definition

(1) $$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t} dt, \quad n > 0.$$

The graph of the integrand for various values of n has already been mentioned. The function represents the area to the right of the origin for various values of the parameter n. The function $\Gamma(n)$ is necessarily positive for all positive values of n, for the integrand is positive throughout the range of integration. The variable t is obviously a dummy.

Note that the index of t is $n - 1$ when the argument of the function is n. If n is less than unity the integral is of the type known as improper since the integrand then tends to infinity at the origin. If n is less than, or equal to, zero the integral fails to converge.

We achieve a first result by equating n to unity. This gives

$$\text{(2)} \qquad \Gamma(1) = \int_0^\infty e^{-t}\,dt = 1.$$

If we integrate by parts we have

$$\int t^{n-1}e^{-t}\,dt = \frac{t^n}{n}\,e^{-t} + \int \frac{t^n}{n}\,e^{-t}\,dt.$$

On inserting the limits we can write

$$n\int_0^\infty t^{n-1}e^{-t}\,dt = \Big[t^n e^{-t}\Big]_0^\infty + \int_0^\infty t^n e^{-t}\,dt.$$

The first term on the right has already been proved to vanish at both limits. We thus have

$$\text{(3)} \qquad n\Gamma(n) = \Gamma(n+1).$$

This is known as the difference equation for the Gamma function and it is one of its most important properties. We proceed to investigate some of its consequences.

(i) Putting $n = 1$ gives $\Gamma(2) = 1\Gamma(1) = 1$. An application of Rolle's theorem to the two results $\Gamma(2) = 1 = \Gamma(1)$ indicates that $\Gamma(n)$ has a minimum value. Actually it occurs when $n = 1{\cdot}4616 \ldots$ and the corresponding value of $\Gamma(n)$ is $0{\cdot}8856 \ldots\ldots$

(ii) By repeated application when n is an integer, e.g. $n = 5$, we get

$$\Gamma(5) = 4\Gamma(4) = 4\,.\,3\Gamma(3) = 4\,.\,3\,.\,2\Gamma(2) = 4!$$

It is easily seen that in general we have $\Gamma(n) = (n-1)!$ when n is an integer. As a matter of history, the problem of finding a function of x that should be continuous when x is positive and take the value $x!$ when x is an integer was solved by Euler in 1729. The Gamma function can accordingly be regarded as a sort of generalized factorial.

(iii) When n is not an integer, e.g. $n = \frac{7}{3}$, we can ultimately reduce the argument below unity. Thus $\Gamma(\frac{7}{3}) = \frac{4}{3}\Gamma(\frac{4}{3}) = \frac{4}{3}\,.\,\frac{1}{3}\Gamma(\frac{1}{3})$. It follows that for purposes of computation the function need only be tabulated in the range 0 to 1. It will appear later that even this range can be reduced.

(iv) By an uncritical use of the difference equation we can carry the function over into negative values of the argument. If, for example, we put $n = \frac{1}{3}$ in the relation $\Gamma(n) = (n-1)(n-2)\Gamma(n-2)$ we get $\Gamma(\frac{1}{3}) = (-\frac{2}{3})(-\frac{5}{3})\Gamma(-\frac{5}{3})$ whence $\Gamma(-\frac{5}{3})$ is defined as $\frac{9}{10}\Gamma(\frac{1}{3})$. It should be noted that the integral definition no longer holds good since the integral fails to converge at the lower limit when $n = -\frac{5}{3}$.

With negative integral values we encounter a series of infinities. Thus

$$\Gamma(2) = 1 \,.\, \Gamma(1) = 1 \,.\, 0 \,.\, \Gamma(0) = 1 \,.\, 0 \,.\, -1 \,.\, \Gamma(-1)$$
$$= 1 \,.\, 0 \,.\, -1 \,.\, -2 \,.\, \Gamma(-2)$$

and so on. It follows that $\Gamma(-n)$ is infinite if n is a positive integer or zero.

The integral defining the Gamma function can be given various forms by changing the dummy variable. The most useful of these is the substitution $t = x^2$, $dt = 2x\,dx$. We then have

$$\Gamma(n) = 2\int_0^\infty x^{2n-1}e^{-x^2}dx. \tag{4}$$

With the particular value $n = \frac{1}{2}$ we recover the error function, which leads to

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-x^2}dx.$$

Hence the important result

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \tag{5}$$

It is now possible to form an idea of the graph of the function, which is indicated in fig. 3. In virtue of the difference equation the value tends to infinity with increasing n, and also when n tends to zero. The reader should satisfy himself that each branch on the left lies wholly on one side of the axis, and the stationary values gradually approach indefinitely close to the axis, though the first two are considerably greater in absolute value than the stationary value on the right. Incidentally the stationary values do not occur at corresponding points; nor are the branches on the left symmetrical, as they are in the graph of $\sec x$. It is an interesting exercise to prove that the equation $\Gamma(n) = c$ has an infinite number of roots for all values of c, positive and negative. The Gamma function has no real zeros except at negative infinity.

The name and notation of the Gamma function are due to Legendre. It is as well to point out, since the reader is likely to consult books other than this, that an alternative function is sometimes used, known as Gauss's Π func[illegible] It only needs to be remembered that the difference is one of [illegible]on only and that $\Pi(n) = \Gamma(n + 1)$, so that for positive integral [illegible] we have $\Pi(n) = n! = \Gamma(n + 1)$.

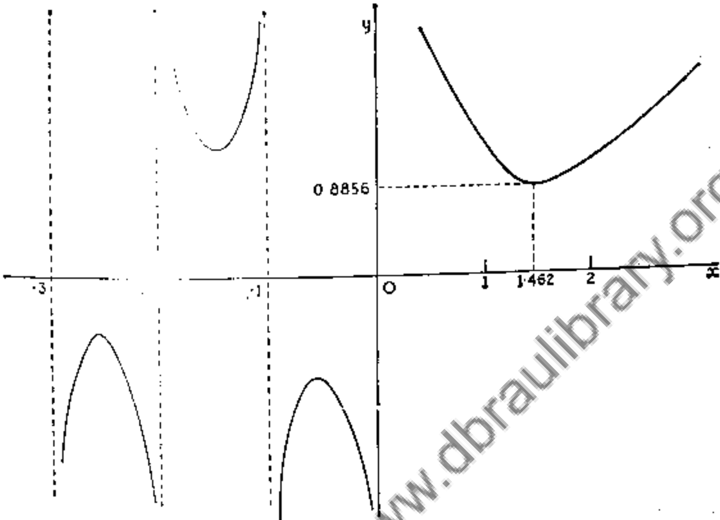

Fig. 3.—Graph of the function $y = \Gamma(x)$

1·5. The Beta function.

The Beta function is one of the class that is defined by a definite integral. It is sometimes known as the First Eulerian integral and it involves two positive parameters which we may take to be p, q. The definition is

$$(1) \qquad B(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1} dt, \quad p, q > 0.$$

The reader should sketch the possible types of integrand for different values of p and q; but a graph of the Beta function itself is not possible since it involves three-dimensional representation. The function represents the area between the ordinates at $t = 0$ and $t = 1$, and the variable t is obviously a dummy. If either p or q, or both are less than unity, the integral becomes improper at one or both limits; and if p or q, or both are negative or zero the integral fails to converge. The integral can be evaluated in finite terms if q is an integer; expansion by the binomial theorem permits termwise integration, but the result is of no value. It will appear later that the Beta function is too closely allied to the Gamma function to have any great importance on its own account.

1·6. Change of variable.

The substitution $t = 1 - x$, $dt = -dx$ gives

(1) $$B(p, q) = \int_0^1 x^{q-1}(1-x)^{p-1}\,dx = B(q, p),$$

so that the function is symmetrical in its two parameters.

A more fruitful substitution is

$$t = \sin^2\theta,\ \ 1 - t = \cos^2\theta,\ \ dt = 2\sin\theta\cos\theta\,d\theta.$$

This gives

(2) $$B(p, q) = 2\int_0^{\frac{1}{2}\pi} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta = B(q, p).$$

This leads to a number of "reduction formulæ" that one encounters when learning the integral calculus. We shall return to these later.

A third useful substitution is $t = y/(1 + y)$. This leads to

$$B(p, q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy.$$

In particular, if $0 < (p, q) < 1$, $p + q = 1$ we have the important result

(3) $$B(1-q, q) = \int_0^\infty \frac{y^{q-1}}{1+y}\,dy = \frac{\pi}{\sin q\pi}.$$

This last integral is a well-known result, easily established by contour integration. Unfortunately, elementary proofs are not so simple. They can usually be found in advanced works on the calculus; in particular, see Edwards, *Integral Calculus*, Vol. 2, p. 61.

Example.—Consider the integral

$$I = \int_0^{\frac{1}{2}\pi} \sqrt{\cot\theta}\,d\theta = \int_0^{\frac{1}{2}\pi} \cos^{\frac{1}{2}}\theta\sin^{-\frac{1}{2}}\theta\,d\theta.$$

Here we have $2p - 1 = -\frac{1}{2}$, $2q - 1 = \frac{1}{2}$, whence $p = \frac{1}{4}$, $q = \frac{3}{4}$. This gives

$$I = \tfrac{1}{2}B(\tfrac{1}{4}, \tfrac{3}{4}) = \frac{\pi}{2\sin\frac{1}{4}\pi} = \frac{\pi}{\sqrt{2}}.$$

1·7. Connexion between B and Γ functions.

The two forms 1·4(4)

$$\Gamma(p) = 2\int_0^\infty x^{2p-1}e^{-x^2}\,dx;\quad \Gamma(q) = 2\int_0^\infty y^{2q-1}e^{-y^2}\,dy$$

lead to

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty\int_0^\infty e^{-x^2-y^2}x^{2p-1}y^{2q-1}\,dx\,dy.$$

On converting to polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$ we can write

$$\Gamma(p)\Gamma(q) = 4\int_0^{\frac{1}{2}\pi}\int_0^{\infty} e^{-r^2}r^{2p+2q-1}\cos^{2p-1}\theta\sin^{2q-1}\theta\,dr\,d\theta.$$

The reader is left to justify the ignoration of the remainder R by the process used when studying the error function. We now have

$$\Gamma(p)\Gamma(q) = 2\int_0^{\frac{1}{2}\pi}\cos^{2p-1}\theta\sin^{2q-1}\theta\,d\theta \times 2\int_0^{\infty} e^{-r^2}r^{2p+2q-1}\,dr.$$

The first integral is known to be $B(p, q)$ and the second integral is a form of $\Gamma(p+q)$. Hence $\Gamma(p)\Gamma(q) = B(p, q)\Gamma(p+q)$, or as it is usually written

(1) $$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

The result emphasizes the symmetry of $B(p, q)$ in its two parameters. It also explains why the Beta function is of little consequence on its own account. If we put $p + q = 1$ the denominator becomes unity and we have, using 1·6(3),

(2) $$\Gamma(q)\Gamma(1-q) = B(q, 1-q) = \frac{\pi}{\sin q\pi}.$$

In particular, if $p = \frac{1}{2} = q$ we have

$$\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}) = \frac{\pi}{\sin\frac{1}{2}\pi}$$

which is our previous result $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

It appears from the result

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

connecting the Gamma functions with the circular functions that for purposes of computation it is necessary to tabulate $\Gamma(n)$ only in the range 0 to $\frac{1}{2}$. Thus $\Gamma(\frac{3}{4})$ is determined from

$$\Gamma(\tfrac{3}{4})\Gamma(\tfrac{1}{4}) = \frac{\pi}{\sin\frac{1}{4}\pi},$$

so that

$$\Gamma(\tfrac{3}{4}) = \frac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})}.$$

A table in this range was calculated to twenty figures by Gauss.

1·8. The duplication formula.

A duplication formula is one which expresses the function of a doubled argument in terms of the function of the single argument. As a simple illustration, the function $\cos 2x$ is expressed in terms of $\cos x$ by the duplication formula $\cos 2x = 2\cos^2 x - 1$.

We have by definition

$$B(n, n) = \int_0^1 x^{n-1}(1-x)^{n-1}\,dx = \int_0^1 (x - x^2)^{n-1}\,dx.$$

The substitution

$$2x - 1 = y,\ dx = \tfrac{1}{2}\,dy$$

gives

$$B(n, n) = \frac{1}{2^{2n-1}}\int_{-1}^1 (1-y^2)^{n-1}dy = \frac{1}{2^{2n-2}}\int_0^1 (1-y^2)^{n-1}dy,$$

since the integrand is symmetrical about the origin. The further substitution

$$y^2 = t,\ dy = \tfrac{1}{2}t^{-\frac{1}{2}}\,dt$$

gives

$$B(n, n) = \frac{1}{2^{2n-1}}\int_0^1 (1-t)^{n-1}t^{-\frac{1}{2}}\,dt = \frac{1}{2^{2n-1}}\,B(n, \tfrac{1}{2}).$$

In terms of Gamma functions this is

$$\frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} = \frac{\Gamma(n)\Gamma(\frac{1}{2})}{2^{2n-1}\Gamma(n+\frac{1}{2})}.$$

Hence

(1) $$\sqrt{\pi}\,.\,\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+\tfrac{1}{2}).$$

As it is no part of our programme to develop the Gamma function beyond what is requisite for the comprehension of the Bessel functions we refrain from pursuing the matter further; but the student to whom the subject is new would be well advised to follow it a little further in some work on advanced calculus.

EXERCISES

1. $\int_0^\infty x^{n-1}e^{-kx}\,dx = \frac{\Gamma(n)}{k^n}, \quad k > 0.$

2. $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{2\pi}{\sqrt{3}}$, $\Gamma(\frac{5}{6})\Gamma(\frac{1}{3}) = \pi^{\frac{1}{2}}2^{\frac{1}{3}}\Gamma(\frac{2}{3})$, $3^{\frac{1}{2}}\{\Gamma(\frac{1}{3})\}^2 = \pi^{\frac{1}{2}}2^{\frac{1}{3}}\Gamma(\frac{1}{6})$.

3. $\int_0^{\frac{1}{2}\pi}\cos^{4/3}\theta\,\sin^{5/3}\theta\,d\theta = \frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{27\sqrt{\pi}}.$

4. If n is a positive integer, $2^n\Gamma(n+\frac{1}{2}) = 1\,.\,3\,.\,5\,\ldots\,(2n-1)\sqrt{\pi}$.

5. Prove that $\int_0^a t^{p-1}(a-t)^{q-1}\,dt = B(p, q)a^{p+q-1}$.

6. Express the product $2\,.\,5\,.\,8\,\ldots\,(3n-1)$ in terms of the Gamma function and generalize the result. [Ans. $3^n\Gamma(n+\frac{2}{3})/\Gamma(\frac{2}{3})$.]

7. Prove the duplication formula for the Gamma function by integrating the $(2n-1)$th power of the identity, $\sin 2\theta = 2\sin\theta\cos\theta$.

8. The sum of two positive numbers is unity. Calculate the root mean square of their product. [Ans. $1/\sqrt{30}$.]

9. By means of the substitution $t = (x-b)/(a-b)$ prove that

$$\int_b^a (x-b)^{p-1}(a-x)^{q-1}\,dx = B(p, q)\,.\,(a-b)^{p+q-1}.$$

10. $\int_0^{\frac{1}{2}\pi}\cos^n\theta\,d\theta = \int_0^{\frac{1}{2}\pi}\sin^n\theta\,d\theta = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}n+\frac{1}{2})}{2\Gamma(\frac{1}{2}n+1)}.$

Verify that when n is an integer, odd or even, these correspond to the "reduction formulæ" of the calculus.

11. If n is a positive non-integer, prove that the sign of $\Gamma(-n)$ is $(-)^r$ where r is the integer next greater than n.

12. $\int_0^{\frac{1}{2}\pi}\sqrt{\cos\theta}\,d\theta = \frac{(2\pi)^{3/2}}{\{\Gamma(\frac{1}{4})\}^2}.$

This result is important since it links the Gamma function with the Elliptic functions; see Whittaker and Watson, *Modern Analysis*, p. 524.

CHAPTER II

Differential Equations

2·1. The linear equation.

The next step in our progress towards the study of the Bessel functions is to survey part of the field of ordinary linear differential equations. It is presumed that the reader already has some slight knowledge of the subject and has at least a nodding acquaintance with the standard elementary methods of solution. The object of the present chapter, which is mainly theoretical, is to refresh his memory and to fill in what are probably a few gaps in his equipment. The justification for its inclusion is that we shall ultimately be dealing with differential equations.

The general linear equation may be written

(1) $$py + qy' + ry'' + \ldots = X,$$

where X and the coefficients p, q, r, &c., are in general functions of the independent variable x. The order of the equation is the order of the highest derivative present. The equation is said to be linear as there are no products and powers (other than the first) of the dependent variable y and its derivatives. If X be replaced by zero, the resulting equation is

$$py + qy' + ry'' + \ldots = 0.$$

This is sometimes called the "auxiliary equation"; alternatively, "the reduced equation".

It is a property of the reduced linear equation that if y_1, y_2, y_3, &c., are solutions, so also is $Ay_1 + By_2 + Cy_3 + \ldots$ where A, B, $C \ldots$ are arbitrary constants. The proof is simple; for if y_1 is a solution we have

$$py_1 + qy_1' + ry_1'' + \ldots = 0.$$

Similarly

$$py_2 + qy_2' + ry_2'' + \ldots = 0,$$

$$py_3 + qy_3' + ry_3'' + \ldots = 0,$$

and so on. If we multiply the respective equations by A, B, C . . . and add, putting

$$Ay_1 + By_2 + Cy_3 + \ldots = Y,$$

we have

$$pY + qY' + rY'' + \ldots = 0.$$

The result proves that Y is a solution of the reduced equation. The arbitrary constants are in practice chosen to suit the assigned conditions of a problem.

Suppose that y_1, y_2, &c., are solutions of the reduced equation. They are said to be linearly connected if constants a, b, c, &c., exist, not all zero, such that $ay_1 + by_2 + \ldots \equiv 0$. If no such constants exist, the solutions are said to be linearly independent. It is a property of the reduced equation that the number of linearly independent solutions is equal to the order. If the order is n and the independent solutions are y_1 to y_n, then the most general solution is $Ay_1 + By_2 + \ldots + Hy_n$, where the coefficients are arbitrary constants. This solution is known as the complementary function; it contains arbitrary constants, in number equal to the order.

A solution of (1) other than a form of the complementary function is called a particular integral. The full solution of (1) is the sum of the complementary function and the particular integral. It is easily proved that this sum is a solution; for if Y is the complementary function we have

$$pY + qY' + rY'' + \ldots = 0,$$

and if Z is the particular integral we have

$$pZ + qZ' + rZ'' + \ldots = X.$$

Addition gives

$$p(Y + Z) + q(Y' + Z') + \ldots = X,$$

which proves that $Y + Z$ is a solution of (1).

2·2. Constant coefficients.

The reader is probably familiar with the case where the coefficients p, q, r . . . in 2·1(1) are constants. The reduced equation is then solved by the substitution $y = e^{mx}$ which leads to

$$(p + qm + rm^2 + \ldots)e^{mx} = 0.$$

The bracket has a number of zeros m_1, m_2, . . . and the corresponding solutions are e^{m_1x}, e^{m_2x}, &c., so that the complementary function is

$$Y = Ae^{m_1x} + Be^{m_2x} + \ldots .$$

There are rules for dealing with repeated values of m, as also for conjugate complex or imaginary values. There are other rules for determining the particular integral in the commonly occurring cases, such as when X is polynomial, exponential or trigonometrical. These things are not our immediate concern; they can be revised in texts specially devoted to the subject.

2·3. Variable coefficients.

When the coefficients in 2·1(1) are no longer constants the outlook is almost completely changed. There are no longer any rules for finding the particular integral, and if one meets an equation which is not reduced it is usually a matter of trial or guesswork to get the particular integral. Fortunately, most of the equations which one encounters are reduced and the need for finding a particular integral does not arise.

As for the reduced equation, there is no golden rule for solving it. Relatively few equations are soluble in finite terms, and in fact the vast majority of differential equations remain insoluble. A number of methods are available; they include solution in series, solution by definite integrals and by contour integration. We shall return to the first two methods later. It should be borne in mind that in general a differential equation defines a transcendental function, and the study of such a function is not altogether precluded by our inability to solve the equation in finite terms. In any case a number of quite general propositions can be enunciated; they include:

(i) If a solution of the reduced equation is known, the order can be lowered. In practice the known solution is often found by trial or inspection.

(ii) The number of linearly independent solutions of the reduced equation cannot exceed the order.

(iii) Linearly independent solutions of the reduced equation exist in number equal to the order.

Further comment will be confined to equations of the second order.

2·4. Second order equations.

As a matter of convenience we take the linear equation of the second order in the form

$$y'' + y'f(x) + yg(x) = h(x).$$

We now propose to establish the first proposition, that "if a solution of the reduced equation is known, the order can be lowered". The proposition is true for any order. In the case of the second order the lowering is to the first order, so that usually the equation is completely soluble in these circumstances.

We accordingly suppose that $u(x)$ is a solution of the reduced equation, so that

$$u'' + u'f + ug = 0.$$

The substitution $y = uv$ implies

$$y' = u'v + uv',$$
$$y'' = u''v + 2u'v' + uv''.$$

The original equation thus transforms to

$$v(u'' + u'f + ug) + v'(2u' + uf) + uv'' = h.$$

The first bracket on the left is zero by hypothesis. Accordingly, if we put $v'(x) = t(x)$ we can write

$$t\left(f + \frac{2u'}{u}\right) + t' = \frac{h}{u}.$$

As this is a first order equation in t, our statement is verified as to lowering. The value of t can be found by the use of an integrating factor, so that v and y can be found in succession.

Example.—Consider the equation

$$y'' + xy' - y = ax^2.$$

The reduced equation

$$y'' + xy' - y = 0$$

is easily seen to have the solution $y = x$, hence we make the substitution $y = xv$. The original equation now takes the form

$$v'' + \left(x + \frac{2}{x}\right)v' = ax.$$

The integrating factor is

$$G = \exp\int\left(x + \frac{2}{x}\right)dx = x^2e^{\frac{1}{2}x^2}.$$

This enables us to write

$$v''G + v'G' = axG = \frac{d}{dx}(v'G),$$

whence

$$v' = G^{-1}\int axG\,dx + bG^{-1} = a\left(1 - \frac{2}{x^2}\right) + \frac{b}{x^2}e^{-\frac{1}{2}x^2}.$$

Here b is an arbitrary constant; and a further integration gives v, so that

$$y = xv = a(x^2 + 2) + cx + bx\int x^{-2}e^{-\frac{1}{2}x^2}\,dx.$$

The second arbitrary constant is c; the integral in the last term cannot be evaluated in finite terms, which is in line with our remark that in general a differential equation defines a transcendental function. It will be observed that the first term on the right corresponds to the particular values $b = 0 = c$ and it is left to the reader to verify that it is the particular integral and really is a solution. The second term is our original guess at a solution of the reduced equation.

We now come to our second proposition. It takes the form: "The reduced linear equation of the second order cannot have more than two linearly independent solutions". It is established by showing that a contrary assumption leads to a contradiction.

We may remind the reader that two simple simultaneous equations may be inconsistent. Thus the two equations

$$4x + 6y + 7 = 0 = 6x + 9y + 5$$

cannot be simultaneously true, since $(6x + 9y)$ is fifty per cent greater than $(4x + 6y)$. If we remedy the deficiency by replacing 5 by $10\frac{1}{2}$, the equations become consistent but are effectively one and the same. In this case there is a value of y for every arbitrarily assigned value of x and the number of solutions is infinite.

Reverting to our reduced differential equation,

(1) $$y'' + y'f(x) + yg(x) = 0,$$

we assume that there can be three linearly independent solutions, which we may denote by y_1, y_2 and y_3. We then have

(2) $$y_1'' + y_1'f + y_1g = 0,$$

(3) $$y_2'' + y_2'f + y_2g = 0,$$

(4) $$y_3'' + y_3'f + y_3g = 0.$$

Multiply the first by λ, the second by μ and add all three. It must be possible to assign λ, μ so that the coefficients of f and g become zero, i.e.

(5) $$\lambda y_1 + \mu y_2 + y_3 = 0,$$

(6) $$\lambda y_1' + \mu y_2' + y_3' = 0.$$

To deny that λ, μ can thus be found is to assert that

$$\frac{y_1'}{y_1} = \frac{y_2'}{y_2}.$$

The integration of this small differential equation leads to

$$\log y_1 = \log y_2 + \log c; \quad y_1 = cy_2.$$

This violates the assumption that the solutions are linearly independent; hence λ and μ can be found. But note that so far there is no justification for thinking them to be constants.

We are also in possession of the further equation

$$\lambda y_1'' + \mu y_2'' + y_3'' = 0, \tag{7}$$

which follows automatically from the others. The differentiation of equation (5) gives

$$\lambda y_1' + \lambda' y_1 + \mu y_2' + \mu' y_2 + y_3' = 0,$$

so that on subtracting (6) we have

$$\lambda' y_1 + \mu' y_2 = 0.$$

Similarly from (6) and (7) we derive

$$\lambda' y_1' + \mu' y_2' = 0.$$

In the last two equations the possibility

$$\frac{y_1'}{y_1} = \frac{y_2'}{y_2}$$

is ruled out; hence we must have

$$\lambda' = 0 = \mu',$$

so that λ and μ are absolute constants. This establishes that the three solutions are in fact connected by a linear relation

$$\lambda y_1 + \mu y_2 + y_3 = 0.$$

It is one thing to say that there cannot be more than two independent solutions, but it is quite a different matter to prove that two independent solutions actually exist. A proposition of the latter type is known as an "existence theorem". Existence theorems usually have three properties; they are long, dull and difficult. It takes a special type of mind, even a special type of mathematical mind, to revel in existence theorems; and as the reader is presumed not to be a mathematical specialist he may be condoned if he accepts the proposition on higher authority. We shall accordingly assume that, except perhaps in certain artificial and factitious cases, the second order equation has two independent solutions which are continuous over certain ranges limited by the infinities of the coefficients.

Parenthetically, let y_1, y_2 denote the two independent solutions; then the general solution is $y = Ay_1 + By_2$ where A, B are arbitrary constants that can be chosen to fit assigned conditions. This raises the question of what two solutions are fundamental. The answer is purely a matter of choice or convenience. Consider the oscillation equation $\ddot{x} + \omega^2 x = 0$ for simple harmonic motion. It has the solutions

$$x = e^{i\omega t},\ e^{-i\omega t},\ \cos\omega t,\ \sin\omega t,\ 4\sin\left(\omega t + \frac{\pi}{6}\right),\ 5\cos\left(\omega t - \frac{\pi}{3}\right)$$

and so on. None of these is more fundamental than any other, and a linear combination of any two is a solution. With other differential equations there is usually some property that makes one form more desirable than another, e.g. it may vanish at the origin, or be more easily calculable. It results that occasionally mathematicians differ as to what are the best forms to employ, and the reader must be prepared to encounter this diversity.

2·5. Properties of the independent solutions.

It has already been mentioned that our inability to solve a particular equation does not wholly deter us from studying the function which it defines. The second order equation, even when insoluble, can be made to give information of two types. The first concerns relations between the two solutions and the second concerns properties of their zeros. We begin with the former.

Suppose that y_1, y_2 are two independent solutions of equation 2·4(1), so that equation 2·4(2) and 2·4(3) hold. On eliminating g we have

$$(y_1''y_2 - y_1y_2'') + (y_1'y_2 - y_1y_2')f = 0.$$

Note that the first bracket is the derivative of the second, so that if we denote the latter by z we can write $z' + zf = 0$. The integrating factor is $G = \exp \int f(x)\,dx$, so that

$$\frac{d}{dx}(zG) = 0, \quad z = AG^{-1},$$

or

(1) $$y_1'y_2 - y_1y_2' = A \exp\{-\int f(x)\,dx\},$$

an extremely useful relation.

Example 1.—Consider the equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$

It is known as Legendre's equation, and the two transcendental functions which are its linearly independent solutions corresponding to our y_1, y_2 are denoted by P_n and Q_n. Here we have

$$f(x) = \frac{-2x}{1-x^2} = \frac{d}{dx}\log(1-x^2).$$

Hence $G = (1 - x^2)$ and the relation becomes

$$P_nQ_n' - P_n'Q_n = \frac{A}{(1-x^2)}.$$

Here A is an absolute constant and its actual value will depend on the forms adopted for P_n and Q_n.

The relation (1) can be carried a step further; for we have

$$\frac{y_1'y_2 - y_1y_2'}{y_2^2} = \frac{A}{Gy_2^2} = \frac{d}{dx}\left(\frac{y_1}{y_2}\right).$$

Hence on integration,

(2) $$y_1 = y_2\left(B + A\int\frac{dx}{Gy_2^2}\right).$$

It follows that if y_2 is known, y_1 can be determined, as stated previously.

Example 2.—Using, as before, the reduced equation

$$y'' + xy' - y = 0,$$

we have

$$f(x) = x, \quad \int f(x)\,dx = \tfrac{1}{2}x^2,$$

whence $G = e^{\frac{1}{2}x^2}$. As $y_2 = x$ we derive

$$y_1 = x(B + A\int x^{-2}e^{-\frac{1}{2}x^2}dx),$$

which agrees with our previous result.

2·6. Zeros of the solutions.

We come now to the second type of information. It concerns the zeros of the functions, the values of x for which the function vanishes, or the places where the graph crosses the x-axis. And lest the reader think we rather harp on these zeros, it may mollify him to know that they usually hold the key to any problem under discussion.

If we take the reduced equation in a form where the coefficients are devoid of fractions we can write

$$y''p(x) + y'q(x) + yr(x) = 0. \tag{1}$$

The statement that the general solution $Ay_1 + By_2$ can be made to fit two conditions has one curious limitation which we shall later find useful. It may be stated thus: No solution of (1) can touch the x-axis except at a zero of $p(x)$.

We are assuming that p, q and r are finite and have derivatives of all orders. If the solution touches the x-axis we have $y = 0 = y'$, whence, if $p(x)$ is not zero, $y'' = 0$ by the equation. Differentiation gives

$$y'''p + y''(p' + q) + y'(q' + r) + yr' = 0,$$

so that if $p(x)$ is not zero it follows that $y''' = 0$ and all the succeeding derivatives can be similarly shown to vanish. The application of Taylor's theorem to the calculation of the neighbouring ordinate gives

$$y = f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots.$$

Since every term in the series vanishes by hypothesis, the ordinate must vanish identically and the curve degenerates to the x-axis.

As an example, revert to Legendre's equation, where $p(x) = (1 - x^2)$. No solution can touch the x-axis except at the two points given by $x^2 = 1$. It is a waste of time seeking a solution that touches elsewhere.

2·7. The normal form.

In the study of algebraic equations it is often an advantage to remove certain terms. It is similarly found advantageous in certain theoretical investigations to remove the middle term from the reduced second order differential equation. The result is known as the normal form and it can be reached by the substitution $y = uv$. Taking the equation in the form

$$y'' + y'f(x) + yg(x) = 0,$$

this gives as before

$$v(u'' + u'f + ug) + v'(2u' + uf) + uv'' = 0.$$

The coefficient of v' vanishes if we choose u so that

$$2u' + uf = 0, \quad \frac{u'}{u} + \tfrac{1}{2}f = 0, \quad \log u + \tfrac{1}{2}\int f(x)\,dx = 0;$$

$$u = \exp\{-\tfrac{1}{2}\int f(x)\,dx\}, \quad u' = -\tfrac{1}{2}uf, \quad u'' = -\tfrac{1}{2}uf' + \tfrac{1}{4}uf^2.$$

The equation becomes

$$\frac{d^2v}{dx^2} + vI = 0,$$

where

$$I(x) = g - \tfrac{1}{2}\frac{df}{dx} - \tfrac{1}{4}f^2, \quad v = \frac{y}{u}.$$

It will be observed that the normal form is more comprehensive than the original whence it derives. Any number of equations may have the same normal form. For a given I we can assign f arbitrarily and then g follows automatically.

Example.—As an illustration, consider the reduced equation

$$y'' + xy' - y = 0.$$

This has

$$g = -1, \quad f = x, \quad u = \exp(-\tfrac{1}{4}x^2), \quad I = -(x^2 + 6)/4.$$

Hence the normal form is

$$\frac{d^2v}{dx^2} = \tfrac{1}{4}v(x^2 + 6), \quad v = y\exp(\tfrac{1}{4}x^2).$$

It may be added that the normal form is rarely an aid to solution; its utility lies in other directions, principally in connexion with the zeros. The following is a fair sample of the type of argument. If a and b be two permissible limits of integration, we can integrate the normal form

$$v'' + vI = 0$$

to give

$$\Big[v'\Big]_b^a + \int_b^a vI\,dx = 0.$$

If in particular a and b are two consecutive positive zeros of the derivative v' we have

$$\int_b^a vI\,dx = 0,$$

the obvious implication being that the integrand cannot maintain sign in the range $b < x < a$. If it were further known that I maintained sign in this interval, we should be driven to the conclusion that

v must change sign. In other words, v must in these circumstances have a zero between two consecutive zeros of its derivative. All of which, of course, depends on the derivative having two zeros; it may have none. The converse theorem, known as Rolle's theorem, is encountered early in one's study of the calculus.

Example.—Consider the equation

$$y'' + 2y' \sin x + y(1 + \sin^2 x) = 0.$$

The hope of solving it is rather remote; but it is easily shown to have the normal form

$$v'' + v(1 - \cos x) = 0.$$

Here I does not change sign in any range. It follows that if any of the infinite number of solutions of the equation for v has maxima and minima, these must alternately lie on opposite sides of the x-axis. We also have

$$u = \exp\left[-\tfrac{1}{2}\int 2 \sin x\, dx\right] = \exp(\cos x),$$

which vanishes for no value of x. As $y = uv$, the zeros of y are governed by the zeros of v.

2·8. No common zero.

Two independent solutions of the normal form cannot have a common zero, or their graphs can never cross the x-axis at the same place. We establish this interesting and important result by assuming that v_1 and v_2 are independent solutions of

$$v'' + vI = 0.$$

From the equations

$$v_1'' + v_1 I = 0 = v_2'' + v_2 I$$

we deduce

$$v_1'' v_2 - v_1 v_2'' = 0 = \frac{d}{dx}(v_1' v_2 - v_1 v_2'),$$

whence

$$v_1' v_2 - v_1 v_2' = A.$$

If v_1 and v_2 can be simultaneously zero, then the constant A must be zero. In that case we could write

$$\frac{v_1'}{v_1} = \frac{v_2'}{v_2},$$

whence $v_1/v_2 = \text{const.}$, which violates the assumption that v_1, v_2 are independent. We conclude that v_1, v_2 have no zero in common.

2·9. Interlacing of zeros.

We can now prove the interesting and important theorem that the zeros of v_1 and v_2 interlace. Having already established that

$$v_1'v_2 - v_1v_2' = A$$

we can write

$$\frac{v_1'v_2 - v_1v_2'}{v_2^2} = \frac{A}{v_2^2} = \frac{d}{dx}\left(\frac{v_1}{v_2}\right),$$

whence

$$\left[\frac{v_1}{v_2}\right]_b^a = \int_b^a \frac{A}{v_2^2}\, dx.$$

Choose a, b to be two consecutive zeros of v_1, if such exist. We already know that v_2 cannot vanish at either end-point; let us further assume that it nowhere vanishes in the range $a > x > b$. Then v_1/v_2 is a continuous function that vanishes at both end-points, so that the value of the above integral is zero as calculated from the left. But this establishes a contradiction, for the integrand does not even change sign in the range. We conclude that v_1/v_2 is not a continuous function; in other words, v_2 has a zero between the consecutive zeros of v_1 at a and b. It is immaterial which of the solutions we call v_1 and v_2, so that each function must have a zero between two consecutive zeros of the other; the zeros of either occur alternately.

In case the reader finds the matter abstruse, let him consider the very ordinary normal form

$$\frac{d^2y}{dx^2} + \omega^2 y = 0.$$

It has the two independent solutions $\sin\omega x$, $\cos\omega x$. These evidently have no common zero; and their zeros interlace. For the matter of that, the same is true of the two independent solutions $5\sin(\omega x - \frac{1}{3}\pi)$, $7\cos(\omega x + \frac{1}{4}\pi)$.

2·10. A comparison theorem.

Before proceeding further it is necessary to recall a mean-value theorem from the integral calculus. It concerns the integral of the product of two functions and shows how this might be related to the integral of one of the functions alone. Working in some definite range of values $a > x > b$, we stipulate that the first function $f(x)$ shall be one-valued and continuous. Somewhere in the range is a value of x, which we may call ξ, where $f(x)$ takes the value $f(\xi)$. With the second function $\phi(x)$ we are not concerned about its continuity; but we insist that it shall be one-valued and not change sign in the range. The theorem states that

$$\int_b^a f(x)\phi(x)\,dx = f(\xi)\int_b^a \phi(x)\,dx, \quad a > \xi > b.$$

The proof is simple, for the continuity of $f(x)$ guarantees that it achieves its upper bound U and its lower bound L. As $\phi(x)$ does not change sign, the value of the integral on the left must lie between

$$U\int_b^a \phi(x)\,dx \quad \text{and} \quad L\int_b^a \phi(x)\,dx.$$

Hence there is a number K between U and L for which we can write

$$\int_b^a f(x)\phi(x)\,dx = K\int_b^a \phi(x)\,dx, \quad U > K > L.$$

The continuity of $f(x)$ further guarantees that it takes every value between U and L at least once. Hence there must be a value of x, which we have called ξ, for which $f(\xi) = K$. This establishes the result.

We now revert to our differential equations. The normal form becomes most fruitful when comparison is made with a similar and simpler form. Suppose that we have two normal forms

$$u'' + uH = 0 = v'' + vK.$$

From these we derive

$$uv'' - u''v = (H - K)uv,$$

whence by integration

$$\Big[uv' - u'v\Big]_b^a = \int_b^a (H - K)uv\,dx.$$

Let a, b be two consecutive positive zeros of v; there is no loss of generality in assuming that v is positive when $a > x > b$. This makes v' positive at the left end and negative at the right, as a rough sketch will show. We now have

$$u(a)v'(a) - u(b)v'(b) = \int_b^a (H - K)uv\,dx,$$

and we propose to show that if $H > K$ throughout the range $a > x > b$ then u must have a zero between a and b. For since $(H - K)v$

is presumed not to change sign, being in fact positive, and $u(x)$ is presumed to be one-valued and continuous, we can write

$$u(a)v'(a) - u(b)v'(b) = u(\xi)\int_b^a (H - K)v\,dx, \quad a > \xi > b.$$

Inspection shows that $u(a)$, $u(b)$ and $u(\xi)$ cannot all have the same sign, which proves that $u(x)$ must somewhere cross the x-axis and have a zero in the range $a > x > b$.

As a corollary, if we assume that u and v have a common zero at $x = b$, the relation runs

$$u(a)v'(a) = u(\xi)\int_b^a (H - K)v\,dx.$$

As the integrand is positive and $v'(a)$ is negative, it follows that $u(a)$ and $u(\xi)$ are of opposite sign. Hence $u(x)$ vanishes a second time before $v(x)$ has time to vanish at $x = a$.

Example.—Consider the equation $y'' + x^{-1}e^x y = 0$, of which the solution is by no means obvious; in fact, its solution is not easy to obtain. Away from the origin, the coefficient $x^{-1}e^x$ is continuous and has derivatives of all orders. Compare this with the equation $y'' + n^2y = 0$, the zeros of whose solutions occur at intervals of π/n. Whatever the value of n, it must eventually happen for sufficiently large values of x that $x^{-1}e^x$ exceeds n^2, so that $(x^{-1}e^x - n^2)$ is positive. It follows that any solution of the given equation must ultimately have a zero in every range whose length is π/n. With increasing x we can afford to make n larger in the comparison equation, whence we conclude that the interval between consecutive zeros of solutions of the given equation probably decreases with increasing x.

2·11. Equations containing a parameter.

Having expounded all that we shall need to know of the equation in its normal form, we turn to another matter that will figure prominently in the ensuing pages. It sometimes happens that a differential equation contains a parameter n; the case of Legendre's equation has already been mentioned, as also the oscillation equation $y'' + n^2y = 0$. The function of x defined by the equation then differs in value according to the value of n. The symbol denoting the function has the parameter written as a suffix and this is called the "order" of the function. In such cases there are usually simple relations connecting functions of contiguous orders. These relations are known as "recurrence formulæ" and are particularly useful in computation.

As an example, the equation $xy'' + (1 - x)y' + ny = 0$ is known as Laguerre's equation of order n and its solution is denoted by $L_n(x)$. Its recurrence formula is known to be

$$L_{n+1}(x) - (2n + 1 - x)L_n(x) + n^2L_{n-1}(x) = 0,$$

and this connects the functions of the three contiguous orders $n + 1$, n and $n - 1$. In any given case the recurrence formula can be derived only when the solution of the differential equation is known. Conversely, the differential equation cannot be derived from the recurrence formula alone. Usually there is a connecting differential relation; in the present case it happens to be

$$L_n'(x) - nL'_{n-1}(x) = -nL_{n-1}(x).$$

2·12. Orthogonal functions.

Two functions which merely differ in their order usually possess interesting integral properties, one such property in particular being known as the orthogonal property. We can illustrate this by Legendre's equation. Let $P_m(x)$, $P_n(x)$ be the functions that respectively satisfy the equations

$$(1 - x^2)P_m'' - 2xP_m' + m(m + 1)P_m = 0.$$
$$(1 - x^2)P_n'' - 2xP_n' + n(n + 1)P_n = 0.$$

On multiplying these respectively by P_n, P_m we deduce

$$\{m(m + 1) - n(n + 1)\}P_nP_m = \frac{d}{dx}\{(1 - x^2)(P_n'P_m - P_nP_m')\}.$$

A bracket on the right suggests limits of integration and we have

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0, \quad m \neq n.$$

This is the orthogonal property, and the functions P_1, P_2 . . . are said to form an orthogonal family. As a matter of fact the reader has for years been familiar with at least two orthogonal families. The one is $\sin x$, $\sin 2x$, $\sin 3x$ . . ., which has the property

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0, \quad m \neq n.$$

The other is obtained on replacing sin by cos.

EXERCISES

1. Solve Laguerre's equation of order unity, viz.

$$xy'' + (1 - x)y' + y = 0,$$

after first verifying that $y = 1 - x$ is a solution.

2. Prove that any two solutions of the equation

$$y'' \cos x + y' \sin x + y \sin^2 x = 0$$

are connected by the relation

$$y_1 y_2' - y_1' y_2 = a \cos x.$$

3. Verify that e^{-x} is a solution of

$$y''(\cos x - \sin x) + 2y' \cos x + y(\cos x + \sin x) = 0,$$

and hence deduce the other solution. In what circumstances could a solution touch the x-axis?

4. Prove that the last equation has the normal form $v'' - v\,\dfrac{1 + \sin^2 x}{1 - \sin 2x} = 0.$

5. Prove that the equation of damped oscillations, $\ddot{x} + 2a\dot{x} + (a^2 + b^2)x = 0$ has the same normal form as when the oscillations are undamped.

6. Reduce the equation $y'' + \dfrac{y'}{x} + \left(1 - \dfrac{1}{4x^2}\right)y = 0$ to normal form. What do you conclude from the result?

7. Prove that any solution of $x^2y'' + xy' + (x^2 - n^2)y = 0$ must have an infinity of zeros.

8. Any solution of $xy'' + 2y' + (1 + x)y = 0$ has an infinity of zeros whose interval ultimately tends to π.

9. Without using the normal form, prove that two independent solutions of the reduced equation cannot have a common zero.

10. If H is greater than K in the two normal forms

$$u'' + uH = 0 = v'' + vK,$$

it was proved in the text that u must have a zero between successive zeros of v. Give a proof without the mean-value theorem. Assume that $u(x)$ has no such root and can without loss of generality be taken as positive throughout the range. Show that this leads to a contradiction.

11. Weber's function $D_n(x)$ has the recurrence formula

$$D_{n+1} - xD_n + nD_{n-1} = 0$$

and satisfies the differential relation

$$D_n' + \tfrac{1}{2}xD_n - nD_{n-1} = 0.$$

Deduce that it is a solution of the differential equation

$$4y'' + y(4n + 2 - x^2) = 0.$$

12. Deduce the recurrence formula for Laguerre's function from the differential equation

$$xy'' + (1 - x)y' + ny = 0$$

and the differential relation

$$L_n' - nL_{n-1}' = -nL_{n-1}.$$

13. Prove that Laguerre's function has the orthogonal property

$$\int_0^\infty e^{-x} L_n L_m \, dx = 0.$$

14. The Tschebyscheff function T_n of order n satisfies the equation

$$(1 - x^2)y'' - xy' + n^2 y = 0.$$

Prove that it has the orthogonal property

$$\int_{-1}^{1} \frac{T_n T_m}{\sqrt{(1 - x^2)}} \, dx = 0.$$

15. The Hermite function $H_n(x)$ of order n satisfies the differential relation

$$H_n' = 2nH_{n-1}$$

and has the recurrence formula

$$H_{n+1} - 2xH_n + 2nH_{n-1} = 0.$$

By differentiating the latter, show that H_n satisfies the differential equation

$$y'' - 2xy' + 2ny = 0.$$

Deduce the orthogonal property

$$\int_{-\infty}^{\infty} H_n H_m e^{-x^2} \, dx = 0.$$

The Hermite and Laguerre functions occur in the applications of Schrödinger's wave equation in quantum theory; see H. Weyl, *The Theory of Groups and Quantum Mechanics*, pp. 54–70; B. L. van der Waerden, *Die gruppentheoretische Methode in der Quantenmechanik*, pp. 12-16.

CHAPTER III

Cylinder Functions

3·1. Recurrence formulæ.

A cylinder function $C_n(x)$ may be defined as a function of x involving a parameter n and satisfying the two recurrence formulæ

$$C_{n-1}(x) + C_{n+1}(x) = \frac{2n}{x} C_n(x),$$

$$C_{n-1}(x) - C_{n+1}(x) = 2 \frac{d}{dx} C_n(x).$$

In general x and n are unrestricted; but in this work attention will be confined to such values of x as are real and positive, with n real but not necessarily integral or positive.

With a view to simplifying the notation, the argument x will be dropped when not in doubt, so that we shall write C_n for $C_n(x)$; and a prime, or dash, will be used to denote differentiation with respect to the argument, so that

$$C_n' = \frac{d}{dx} C_n(x); \quad C_n'(ax) = \frac{d}{d(ax)} C_n(ax) = \frac{1}{a} \frac{d}{dx} C_n(ax).$$

The above two formulæ are accordingly written

(1) $$C_{n-1} + C_{n+1} = \frac{2n}{x} C_n,$$

(2) $$C_{n-1} - C_{n+1} = 2C_n'.$$

By addition and subtraction they are equivalent to

(3) $$xC_{n-1} = nC_n + xC_n',$$

(4) $$xC_{n+1} = nC_n - xC_n'.$$

It will now be shown that such a function C_n must satisfy a certain differential equation, known as Bessel's equation, of which the solutions are Bessel functions. Accordingly the cylinder functions are Bessel functions, or combinations thereof; but as there are at least six different types of Bessel function, each with its own peculiarities and bearing the name of some eminent mathematician, yet all possessing properties in common, there is a certain advantage in retaining the term cylinder function for the moment. We can discriminate between them later on.

Much of mathematical physics is dominated by Laplace's equation and Laplace's operator, and when these are translated into cylindrical co-ordinates, Bessel's equation almost inevitably appears, as will be found more than once in the ensuing pages. This accounts for the name "cylinder function" which is common on the Continent and in America. It appears to be due to Heine.

3·2. Bessel's equation.

The differentiation of 3·1(3) gives

(1) $$xC_n'' + (n+1)C_n' = xC_{n-1}' + C_{n-1}.$$

Multiply 3·1(3) by n and subtract from (1) multiplied by x. This gives

$$x^2C_n'' + xC_n' - n^2C_n = x\{xC_{n-1}' - (n-1)C_{n-1}\} = -x^2C_n,$$

the last step being justified by 3·1(4). The result shows that C_n satisfies the equation

(2) $$x^2y'' + xy' + (x^2 - n^2)y = 0,$$

which is Bessel's equation of order n.

The recurrence formulæ cannot be directly deduced from Bessel's equation and it is irrational to conclude that any particular Bessel function necessarily satisfies the recurrence formulæ. This happens to be true when the arbitrary constants in the solution of (2) are properly chosen as functions of n; but it cannot be substantiated until Bessel's equation has been solved. For the solutions that will be adopted, it will prove easy to verify that 3·1(3) is satisfied, and the rest will follow. For, reversing the above argument, a function that satisfies (2) and 3·1(3) also satisfies (1) and hence it satisfies 3·1(4).

3·3. Interlacing of zeros.

The recurrence formulæ rapidly lead to important deductions. Thus 3·1(3) is equivalent to

(1) $$x^nC_{n-1} = nx^{n-1}C_n + x^nC_n' = \frac{d}{dx}\{x^nC_n\}.$$

According to Rolle's theorem, between two consecutive zeros of a continuous function lies a zero of its derivative. Taking the continuous function to be $x^n C_n$, between two of its consecutive zeros lies a zero of its derivative, i.e. of $x^n C_{n-1}$. If we move away from the origin to avoid possible discontinuities or such zeros as may come from x^n, we conclude that between two consecutive positive zeros of C_n lies a zero of C_{n-1}.

Similarly 3·1(4) can be written

(2) $$x^{-n}C_n' - nx^{-n-1}C_n = -x^{-n}C_{n+1} = \frac{d}{dx}\{x^{-n}C_n\},$$

whence it follows that, except possibly near the origin, a zero of C_{n+1} lies between consecutive zeros of C_n. Changing the order from n to $n-1$, a zero of C_n lies between consecutive zeros of C_{n-1}, and taking this in conjunction with the similar result obtained from (1) we conclude that the zeros of C_n and C_{n-1} interlace. Incidentally this is not to be confused with the similar proposition, proved in the last chapter, concerning two solutions of the same equation.

Note that (1) can be reversed to give

(3) $$\int x^n C_{n-1}(x)\,dx = x^n C_n(x)$$

and (2) can be written

(4) $$\int x^{-n} C_{n+1}(x)\,dx = -x^{-n} C_n(x).$$

3·4. Integral orders.

It seems at first sight reasonable to suppose that functions of integral order might be more tractable than others; but they possess one awkward drawback. Putting $n = 0$ in the recurrence formulæ 3·1(3) and 3·1(4) we have

(1) $$C_0' = C_{-1} = -C_1.$$

It appears that, of the two functions C_1 and C_{-1} of the first order, the one is a mere numerical multiple of the other. We proceed to investigate further. Putting $n = 1, -1$ in 3·1(1) we have

$$xC_0 + xC_2 = 2C_1,$$
$$xC_{-2} + xC_0 = -2C_{-1}.$$

By subtraction,

$$x(C_2 - C_{-2}) = 2(C_1 + C_{-1}) = 0,$$

whence $$C_{-2} = (-1)^2 C_2.$$

Putting $n = 2, -2$ in 3·1(1) we have

$$xC_1 + xC_3 = 4C_2,$$
$$xC_{-3} + xC_{-1} = -4C_{-2}.$$

By addition

$$x(C_3 + C_{-3}) + x(C_1 + C_{-1}) = 4(C_2 - C_{-2}),$$

whence

$$C_{-3} = (-1)^3 C_3.$$

In general,

$$x(C_{n+1} \pm C_{-n-1}) + x(C_{n-1} \pm C_{-n+1}) = 2n(C_n \mp C_n),$$

according as n is even or odd, and it is an easy induction, which can be left to the reader, to prove that

(2) $$C_{-n} = (-1)^n C_n.$$

Bessel's equation, being of the second order, certainly possesses two linearly independent solutions. The important point is that for integral orders the second independent solution is not obtained by merely changing the sign of n. This complicates matters considerably, and later in the book we shall take up the problem of finding the second solution in such cases.

3·5. The normal form and the zeros.

The ideas expounded in the previous chapter can now be applied to Bessel's equation. Its leading coefficient is x^2, whence we conclude that no solution can have a repeated zero except possibly at the origin. Otherwise expressed, C_n can have no repeated positive zero and the graph cannot touch the x-axis except possibly at the origin. Comparing Bessel's equation with

$$y'' + y'f(x) + yg(x) = 0,$$

we have, in 2·7,

$$f(x) = \frac{1}{x}, \quad g(x) = 1 - \frac{n^2}{x^2},$$

$$u = \exp\{-\tfrac{1}{2}\int f(x)\,dx\} = x^{-\frac{1}{2}}.$$

$$I = g - \tfrac{1}{2}\frac{df}{dx} - \tfrac{1}{4}f^2 = 1 + \frac{1 - 4n^2}{4x^2},$$

(1) $$v'' + v\left(1 + \frac{1 - 4n^2}{4x^2}\right) = 0, \quad v = x^{\frac{1}{2}}C_n(x).$$

This is the normal form of Bessel's equation.

In the particular case where $n = \pm\frac{1}{2}$ the normal form reduces to the simple oscillation equation $v'' + v = 0$ with the solution $v = R \sin(x + \alpha)$. This connects Bessel functions with trigonometrical functions and we have the remarkable result that $x^{\frac{1}{2}}C_{\frac{1}{2}}$ and $x^{\frac{1}{2}}C_{-\frac{1}{2}}$ are of the form $a \cos x + b \sin x$ with an infinite number of zeros at intervals of π.

More generally, I is necessarily positive if (i) $4n^2 < 1$; or (ii) $4x^2 + 1 > 4n^2$, which for any value of n must be the case ultimately for large x. If $n^2 < \frac{1}{4}$, $I > 1$ and is monotonic decreasing to unity. If $n^2 > \frac{1}{4}$, $I < 1$ and is monotonic increasing to unity.

We now propose to show that if $n^2 < \frac{1}{4}$ the function $x^{\frac{1}{2}}C_n$ has at least one zero in any range $0 < \alpha < x < \alpha + \pi$. The proof consists in showing that the contrary assumption leads to a contradiction. Assuming then that the function has no such zero, there is no loss of generality in taking it to be positive throughout the range. The origin is deliberately avoided since we do not know how the function behaves there.

We make comparison with the function $w = \sin(x - \alpha)$, which has consecutive zeros at α and $\alpha + \pi$. Between these values it is positive and it satisfies the equation $w'' + w = 0$. On the left, w' is positive at α, whilst it is negative at $\alpha + \pi$ on the right. From the normal form $v'' + vI = 0$ we deduce

$$vw'' - v''w = (I - 1)vw,$$

whence

$$\int_{\alpha}^{\alpha+\pi} (I - 1)vw\,dx = \Big[vw' - v'w\Big]_{\alpha}^{\alpha+\pi} = \Big[vw'\Big]_{\alpha}^{\alpha+\pi}$$
$$= v(\alpha + \pi)w'(\alpha + \pi) - v(\alpha)w(\alpha).$$

In accordance with our assumptions the right side is essentially negative; but the integrand is nowhere negative in the range. This establishes the contradiction and we conclude that $x^{\frac{1}{2}}C_n$ has a zero in the given range, and hence C_n has an infinite number of positive zeros at intervals of not more than π, provided $n^2 < \frac{1}{4}$.

An upper limit to the interval between the zeros being π when $n^2 < \frac{1}{4}$, we can now establish a lower limit. Let α be a zero of C_n and let

$$I(\alpha) = 1 + \frac{1 - 4n^2}{4\alpha^2} = k^2.$$

Then $I(x) < k^2$ if $x > \alpha$. Consider the function $w = \sin k(x - \alpha)$

which has consecutive zeros at α, $\alpha + \pi/k$ and satisfies the equation $w'' + k^2w = 0$. Then

$$\int_\alpha^\beta (k^2 - I)vw\,dx = \int_\alpha^\beta (wv'' - w''v)\,dx = \Big[wv' - w'v\Big]_\alpha^\beta.$$

Take β to be the zero of v next following α and suppose that, if possible, it falls short of $\alpha + \pi/k$. This ensures that (vx), $w(x)$ and $k^2 - I(x)$ are all positive (or at least, not negative) between α and β. The integral is therefore positive; moreover,

$$v(\beta) = 0 = v(\alpha) = w(\alpha).$$

The right side accordingly reduces to $w(\beta)v'(\beta)$ where $v'(\beta)$ is negative. This makes $w(\beta)$ negative, thus showing that w has crossed the x-axis and contradicting the assumption that β fell short of $\alpha + \pi/k$. We conclude that $\beta > \alpha + \pi/k$, or

(2) $$\alpha + \pi > \beta > \alpha + \frac{\pi\alpha}{(\alpha^2 + \frac{1}{4} - n^2)^{\frac{1}{2}}}.$$

Note that with increasing α the interval approaches π from below, provided $n^2 < \frac{1}{4}$.

We are now in a position to state that any cylinder function, of whatever order, has an infinity of zeros. For choosing any value of n such that $n^2 < \frac{1}{4}$ and calling it p, we have established that C_p has an infinity of zeros. These are interlaced by the zeros of C_{p+1}, and these in turn by the zeros of C_{p+2}, and so on. Working in the other direction, the zeros of C_p interlace, and are interlaced by, the zeros of C_{p-1}; and so on.

We shall now establish a theorem concerning the second non-negative zero. If the origin happens to be a zero, the proposition concerns the magnitude of the first positive zero. Let α_1, α_2 be the first two non-negative zeros of C_n, and let β be the zero of C_n' between them. Obviously $\beta < \alpha_2$ and we propose to show that $n < \beta$. Consider the graph of C_n between α_1 and α_2. If it passes through α_1 from below, C_n is positive, the curve rises to a maximum and C_n'' is negative. Alternatively, if it passes through α_1 from above, C_n is negative, the curve sinks to a minimum and C_n'' is positive. In either case C_n and C_n'' are of opposite sign. By hypothesis $C_n'(\beta) = 0$ and if we put $x = \beta$ in Bessel's equation we have

$$\beta^2 C_n''(\beta) + (\beta^2 - n^2)C_n(\beta) = 0.$$

We conclude that $(\beta^2 - n^2)$ is positive and that β is greater than n in absolute value. Hence $|n| < \beta < \alpha_2$. Descriptively this means that as the order rises the graph is more leisurely about reaching the second zero.

It remains to consider the heights and depths to which the graph rises and falls between crossings of the x-axis. We shall prove that the successive maxima and minima decrease in absolute value. Taking the simple case of zero order, Bessel's equation is

$$C_0'' + \frac{1}{x} C_0' + C_0 = 0.$$

Multiplying by $2C_0'$ and integrating between unspecified limits a and b, we have

$$\left[C_0'^2\right]_a^b + 2\int_a^b \frac{1}{x} C_0'^2 dx + \left[C_0^2\right]_a^b = 0.$$

If a and b are taken to be positions of stationary value, not necessarily consecutive, we have

$$2\int_a^b \frac{1}{x} C_0'^2 dx = C_0^2(a) - C_0^2(b).$$

The integral is certainly positive if we are working on the right of the origin, and so $C_0(a)$ is greater than $C_0(b)$ in absolute value.

More generally, we can multiply Bessel's equation of order n by $2C_n'$ and write

$$\frac{2x^2}{x^2 - n^2} C_n'' C_n' + \frac{2x}{x^2 - n^2} C_n'^2 + 2C_n C_n' = 0.$$

As
$$\frac{d}{dx}\left(\frac{x^2}{x^2 - n^2}\right) = \frac{-2xn^2}{(x^2 - n^2)^2},$$

integration by parts gives

$$\left[\frac{x^2}{x^2 - n^2} C_n'^2\right]_a^b + \int_a^b \left\{\frac{2x}{x^2 - n^2} + \frac{2xn^2}{(x^2 - n^2)^2}\right\} C_n'^2 dx + \left[C_n^2\right]_a^b = 0.$$

Taking as before a, b to be positions of stationary value, not necessarily consecutive, we have

$$\int_a^b \frac{2x^3}{(x^2 - n^2)^2} C_n'^2 dx = C_n^2(a) - C_n^2(b).$$

The integrand is necessarily positive if we work on the right of the origin, hence: The absolute value of any maximum or minimum exceeds the absolute value of any succeeding maximum or minimum.

It is an easy deduction from this that there is only one zero of C_n' between two consecutive zeros of C_n.

To sum up what we have learned of the function $C_n(x)$, we have no formula for calculating it; we do not know its value for any particular value of x; nor can we say how it behaves at the origin. On the other hand it has an infinity of positive zeros which occur at nearly equal

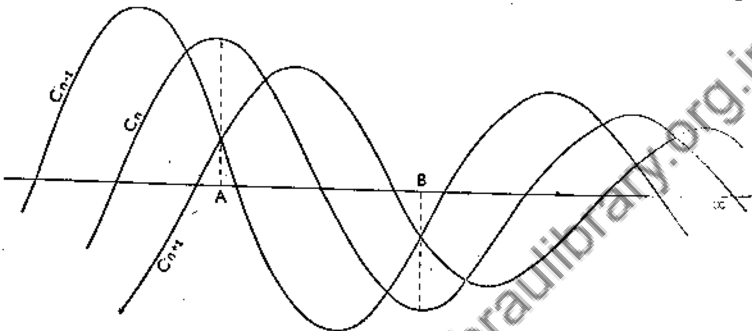

Fig. 4.—March of 3 cylinder functions of consecutive orders
$C_n' = 0$ when $C_{n+1} = C_{n-1}$ (at A and B)

intervals; between a consecutive pair is a turning value whose absolute magnitude steadily decreases, as also a zero of the next consecutive order both above and below. The function of order $\frac{1}{2}$ has its zeros at absolutely equal intervals of π and is expressible in sines and cosines. For the most part the graph looks like a damped vibration, but it cannot be called periodic; for one thing, there is no constant period, and for another thing, the values do not repeat themselves (fig. 4).

3·6. Transformations.

It rarely happens in practice that Bessel's equation makes its appearance in the standard form which we have already given, and it is desirable that one should be in a position to recognize it under other guises. For instance, it is scarcely self-evident that $xy'' + y = 0$ is a form of Bessel's equation.

One of the most obvious forms is obtained on replacing x by kx, so that dx is replaced by $k\,dx$. The equation becomes

(1) $$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(k^2 - \frac{n^2}{x^2}\right)y = 0,$$

with the solution $y = C_n(kx)$. Other exercises are given at the end of the chapter and we append an example of the method.

Example.—We have, as at 3·3(1),

$$\frac{d}{dx}[x^nC_n] = x^nC_{n-1} = x[x^{n-1}C_{n-1}].$$

To find the equation satisfied by x^nC_n we have

$$\frac{d^2}{dx^2}[x^nC_n] = x^{n-1}C_{n-1} + x\frac{d}{dx}[x^{n-1}C_{n-1}]$$

$$= x^{n-1}C_{n-1} + x^2[x^{n-2}C_{n-2}]$$

$$= x^{n-1}C_{n-1} + x^n\left[\frac{2(n-1)}{x}C_{n-1} - Cn\right]$$

$$= (2n-1)x^{n-1}C_{n-1} - x^nC_n.$$

The elimination of C_{n-1} then gives

$$y'' + \frac{1-2n}{x}y' + y = 0$$

as the equation satisfied by $y = x^nC_n$. Note that if $n = \frac{1}{2}$, this reduces to the oscillation equation for simple harmonic motion and again proves that $x^{\frac{1}{2}}C_{\frac{1}{2}}$ can be expressed in terms of $\sin x$ and $\cos x$.

A comprehensive form that covers most cases occurring in practice is

(2) $$y'' + \frac{1-2\alpha}{x}y' + \left[(\beta\gamma x^{\gamma-1})^2 + \frac{\alpha^2 - n^2\gamma^2}{x^2}\right]y = 0,$$

whose solution is

(3) $$y = x^\alpha C_n(\beta x^\gamma).$$

We establish this result as follows. A solution of

$$z^2\frac{d^2t}{dz^2} + z\frac{dt}{dz} + (\beta^2z^2 - n^2)t = 0$$

is $t = C_n(\beta z)$. If $z = x^\gamma$ we have, using dashes to denote differentiation with respect to x,

$$\frac{dt}{dz} = \frac{t'}{z'} = \frac{t'}{\gamma x^{\gamma-1}}.$$

This can be written

$$z\frac{dt}{dz} = \frac{1}{\gamma}x\frac{dt}{dx},$$

whence we get the equivalence of the operators

$$z\frac{d}{dz} \equiv \frac{1}{\gamma}x\frac{d}{dx}$$

when applied to t. The differential equation can be written

$$z\frac{d}{dz}\left(z\frac{dt}{dz}\right) + (\beta^2 z^2 - n^2)t = 0,$$

whence

$$x\frac{d}{dx}\left(x\frac{dt}{dx}\right) + (\beta^2 x^{2\gamma} - n^2)\gamma^2 t = 0.$$

The substitution $t = yx^{-\alpha}$ gives $x\dfrac{dt}{dx} = y'x^{1-\alpha} - \alpha yx^{-\alpha}$,

$$x\frac{d}{dx}\left(x\frac{dt}{dx}\right) = y''x^{2-\alpha} + (1 - 2\alpha)y'x^{1-\alpha} + \alpha^2 yx^{-\alpha}.$$

After multiplying through by $x^{\alpha-2}$ this gives the required result. This will be taken as a standard and will be referred to repeatedly in the following pages.

Example.—As an example of its use, let us settle the question whether the equation $x^2y'' - xy' + (x^4 + \frac{3}{4})y = 0$ is a form of Bessel's equation; and if so, what is the appropriate cylinder function. We recast it as

$$y'' - \frac{y'}{x} + \left(x^2 + \frac{3}{4x^2}\right)y = 0.$$

Comparison with (2) then gives

$$1 - 2\alpha = -1, \quad \beta\gamma = 1,$$

$$2\gamma - 2 = 2, \quad \alpha^2 - n^2\gamma^2 = \tfrac{3}{4};$$

whence

$$\alpha = 1, \quad \beta = \tfrac{1}{2}, \quad \gamma = 2, \quad n = \pm\tfrac{1}{4}.$$

We conclude that the equation is of Bessel's type and has the solutions

$$xC_{\frac{1}{4}}(\tfrac{1}{2}x^2), \quad xC_{-\frac{1}{4}}(\tfrac{1}{2}x^2).$$

It is worth while making a few observations on equation (2). Note that if the middle term is missing, so that the equation is in its normal form, we must have $\alpha = \frac{1}{2}$. This enables us to say at sight that if the middle term is missing, the solution must have the factor $x^{\frac{1}{2}}$. Conversely, a function of the form $x^{\frac{1}{2}}C_n(\beta x^\gamma)$ satisfies an equation in its normal form and y' is missing. Note further that γ cannot be zero; otherwise expressed, the first term in the square bracket cannot be absent. For one thing, the argument of the function ceases to vary and degenerates to a mere constant. Moreover the equation ceases to be of Bessel's type. It degenerates to

$$x^2y'' + (1 - 2a)xy' + a^2y = 0$$

and is soluble by elementary means. The reader can verify that the results are consistent, a solution being $y = x^a$. Incidentally it is illuminating to find the other solution.

A case that occurs often enough to merit special notice is when $\alpha^2 = n^2\gamma^2$, $\gamma = \frac{1}{2}$ so that $\alpha = \frac{1}{2}n$. Its differential coefficients are particularly simple, for if

$$y = x^{\frac{1}{2}n}C_n(\beta x^{\frac{1}{2}})$$

the recurrence formula runs

$$nC_n + \beta x^{\frac{1}{2}}C_n{}' = \beta x^{\frac{1}{2}}C_{n-1},$$

where the unwritten argument is $\beta x^{\frac{1}{2}}$ and the dash denotes differentiation with respect to the argument and not with respect to x. Differentiating a function of a function we have

$$\frac{dy}{dx} = \tfrac{1}{2}nx^{\frac{1}{2}n-1}C_n + \tfrac{1}{2}\beta x^{\frac{1}{2}n-\frac{1}{2}}C_n{}',$$

(4) $$\frac{d}{dx}\{x^{\frac{1}{2}n}C_n(\beta x^{\frac{1}{2}})\} = \tfrac{1}{2}\beta x^{\frac{1}{2}(n-1)}C_{n-1}(\beta x^{\frac{1}{2}}),$$

which might have been derived by change of variable from the previous result

$$\frac{d}{dx}[x^nC_n(x)] = x^nC_{n-1}(x).$$

In both y and its differential coefficient, the degree of the factor x is half the order. Consequently if a point has displacement x at time t given by $x = t^2C_4(kt^{\frac{1}{2}})$ its velocity is $\frac{1}{2}kt^{3/2}C_3(kt^{\frac{1}{2}})$ and its acceleration is $\frac{1}{4}k^2tC_2(kt^{\frac{1}{2}})$.

The companion form, which will figure prominently in the applications later, is

(5) $$\frac{d}{dx}\{x^{-\frac{1}{2}n}C_n(\beta x^{\frac{1}{2}})\} = -\tfrac{1}{2}\beta x^{-\frac{1}{2}(n+1)}C_{n+1}(\beta x^{\frac{1}{2}}).$$

EXERCISES

1. Show that the stationary values of C_0 are located by the zeros of C_1; the stationary values of C_n are located by the roots of $C_{n+1} = C_{n-1}$.

2. Prove (i) $C_2 - C_0 = 2C_0''$; (ii) $C_2 = C_0'' - C_0'/x$; (iii) $C_3 + 3C_0' + 4C_0''' = 0$; (iv) $x^2 C_n'' = xC_{n+1} + \{n(n-1) - x^2\}C_n$.

3. By repeated application of the recurrence formulæ, prove

(i) $$\tfrac{1}{2}xC_{n-1} = nC_n - (n+2)C_{n+2} + (n+4)C_{n+4} - \ldots .$$

(ii) $$\tfrac{1}{2}xC'_n = \tfrac{1}{2}nC_n - (n+2)C_{n+2} + (n+4)C_{n+4} - \ldots .$$

4. Prove that $4C_n'' = C_{n-2} - 2C_n + C_{n+2}$. Hence prove by induction that

$$2^r \frac{d^r}{dx^r} C_n = C_{n-r} - rC_{n-r+2} + \frac{r(r-1)}{2!} C_{n-r+4} - \ldots + (-)^r C_{n+r},$$

where r is a positive integer and the coefficients are binomial.

5. Prove that

$$\frac{d}{dx}[x^n C_n(ax)] = ax^n C_{n-1}(ax),$$

$$\frac{d}{dx}[x^{-n} C_n(ax)] = -ax^{-n} C_{n+1}(ax),$$

$$\frac{d}{dx}[x^{-\frac{1}{2}n} C_n \sqrt{(ax)}] = -\tfrac{1}{2}a^{\frac{1}{2}} x^{-\frac{1}{2}(n+1)} C_{n+1} \sqrt{(ax)}.$$

6. Prove that

$$2\frac{d}{d(x^2)}[x^n C_n] = x^{n-1} C_{n-1},$$

$$(-2)\frac{d}{d(x^2)}[x^{-n} C_n] = x^{-n-1} C_{n+1}.$$

Deduce that

$$2^r \frac{d^r}{d(x^2)^r}[x^n C_n] = x^{n-r} C_{n-r},$$

$$(-2)^r \frac{d^r}{d(x^2)^r}[x^{-n} C_n] = x^{-n-r} C_{n+r},$$

where r is a positive integer.

7. If $Q_n(x) = C_n^2(x)$ prove $Q_{n-1} - Q_{n+1} = \dfrac{2n}{x} Q_n'$.

8. By methods similar to those used in the text, investigate the properties of a function $I_n(x)$ which satisfies the two relations

$$I_{n-1} + I_{n+1} = 2I_n'; \quad I_{n-1} - I_{n+1} = \frac{2n}{x} I_n.$$

9. Justify the following statements when n is positive ;

(i) At any positive zero of C_{n-1}, the functions C_n and C_{n+1} have the same sign.

(ii) At consecutive positive zeros of C_{n-1} the function C_n changes sign.

(iii) Between consecutive positive zeros of C_{n-1} the function C_{n+1} has an odd number of zeros.

Interchanging C_{n-1} and C_{n+1} in the argument, deduce that their zeros interlace. This is an extension of the theorem for consecutive orders.

10. If C_n, K_n are two cylinder functions of the same order (solutions of the same equation), prove from the recurrence formulæ that

$$C_n K_n' - C_n' K_n = C_{n+1} K_n - C_n K_{n+1}.$$

Deduce from Bessel's equation that the common value is A/x, where A is an absolute constant. By considering two consecutive positive zeros of C_n, deduce that the zeros of C_n and K_n interlace. Of what general theorem is this a particular case?

11. Derive the following transforms of Bessel's equation and check by the general formula 3·6(2).

$$\text{(1)}\quad y'' + \frac{1}{x}y' + 4\left(x^2 - \frac{n^2}{x^2}\right)y = 0. \qquad y = C_n(x^2).$$

$$\text{(2)}\quad xy'' + y' + \tfrac{1}{4}\left(1 - \frac{n^2}{x}\right)y = 0. \qquad y = C_n(\sqrt{x}).$$

$$\text{(3)}\quad x^{\frac{1}{2}}y'' + y = 0. \qquad y = x^{\frac{1}{2}}C_{\frac{2}{3}}(\tfrac{4}{3}x^{\frac{3}{4}}).$$

$$\text{(4)}\quad y'' + xy = 0. \qquad y = x^{\frac{1}{2}}C_{\frac{1}{3}}(\tfrac{2}{3}x^{3/2}).$$

$$\text{(5)}\quad xy'' + y = 0. \qquad y = x^{\frac{1}{2}}C_1(2\sqrt{x}).$$

$$\text{(6)}\quad y'' + \frac{2n+1}{x}y' + y = 0. \qquad y = x^{-n}C_n(x).$$

12. If n is half an odd integer, positive or negative, prove by the recurrence formulæ that $x^{\frac{1}{2}}C_n$ has the form $A\cos x + B\sin x$ where A, B are polynomials in x^{-1}.

13. Prove by the use of Bessel's equation that all derivatives of C_{n+1} can be expressed in the form $AC_n + BC_n'$ where A, B are polynomials in x^{-1}. Deduce that, for integral orders, C_n can be similarly expressed as $AC_0 + BC_1$. In particular, work out the results for $n = 2, 3$ and verify that

$$C_4 = \left(1 - \frac{24}{x^2}\right)C_0 + \frac{8}{x}\left(\frac{6}{x^2} - 1\right)C_1.$$

14. If $n^2 > \frac{1}{4}$ and α, β are consecutive zeros of C_n with $0 < \alpha < \beta$, prove that the interval $(\beta - \alpha)$ is at least π.

15. If m is less than n in absolute value, prove that C_m has a zero between consecutive positive zeros of C_n.

16. Prove from Bessel's equation that C_n' cannot have a repeated positive

zero, and that C_n and C_n' cannot have a positive zero in common. Hence from the recurrence relation deduce that C_n and C_{n+1} cannot have a positive zero in common.

17. Prove from Bessel's equation by considerations of sign that C_n' has only one zero between two consecutive zeros of C_n. Deduce further that $axC_n' + bC_n$ has an infinity of positive zeros.

18. If the coefficients a, b are real, prove that $axC_n' + bC_n$ cannot have a repeated positive zero greater than n. [Use the function, and its derivative, in conjunction with Bessel's equation.]

CHAPTER IV

Bessel's Equation

4·1. The series solution.

If we write Bessel's equation of order n in the form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0,$$

with the leading coefficient unity, it appears that the only point where the other coefficients can become infinite (known as the singularities) is the origin. To the practised eye it is then evident from the theory of differential equations (into which we do not propose to enter) that there is a solution in the form of a series of ascending powers of x, convergent for all values of x except possibly at the origin itself.

The formal method of obtaining the series is to assume

$$y = a_o x^r + a_1 x^{r+1} + a_2 x^{r+2} + \dots .$$

If this, with its derivatives, be substituted in the differential equation, the result should be identically zero, so that the coefficient of every power of x vanishes. This procedure supplies a set of recurrence formulæ whereby all the coefficients can be determined in terms of the first, which remains arbitrary. The very first coefficient supplies an equation, known as the "indicial equation", which decides the permissible values of the leading index r.

For the sake of simplicity we shall take Bessel's equation in the modified form

$$xy'' + (2n+1)y' + xy = 0,$$

whose solution is $y = x^{-n}C_n(x)$. We now have

$$\begin{aligned} xy'' &= r(r-1)a_o x^{r-1} + (r+1)ra_1 x^r + \dots , \\ (2n+1)y' &= (2n+1)ra_o x^{r-1} + \dots \qquad\qquad + \dots , \\ xy &= \qquad\qquad\qquad\qquad\qquad\qquad a_o x^{r+1} + \dots . \end{aligned}$$

On taking the sum, the coefficient of x^{r-1} must be zero. Since a_o cannot be zero (the series must start somewhere) we have the indicial equation $r(r + 2n) = 0$, whence $r = 0, -2n$. For preference we shall continue the work using the former value.

We now have

$$y = a_0 + a_1x + a_2x^2 + \dots ,$$

whence

$$\begin{aligned} xy'' &= \qquad\qquad 2a_2x + 3.2a_3x^2 + 4.3a_4x^3 + \dots , \\ xy &= \qquad\qquad a_0x + a_1x^2 + a_2x^3 + \dots , \\ (2n+1)y' &= (2n+1)a_1 + 2(\ )a_2x + 3(\ )a_3x^2 + 4(\ )a_4x^3 + \dots , \end{aligned}$$

where ( ) is written for $(2n + 1)$. This on addition gives $a_1 = 0$ and it appears in succession that every coefficient with odd suffix is zero. Our series is therefore an even function of the form

$$y = a_o + a_2x^2 + a_4x^4 + \dots .$$

Discarding dead matter we now have

$$\begin{aligned} xy'' &= 2.1a_2x + 4.3a_4x^3 + 6.5a_6x^5 + \dots , \\ xy &= a_0x + a_2x^3 + a_4x^5 + \dots , \\ (2n+1)y' &= 2(2n+1)a_2x + 4(2n+1)a_4x^3 + 6(2n+1)a_6x^5 + \dots . \end{aligned}$$

Hence

$$2(2n+2)a_2 = -a_0, \qquad a_2 = -\frac{a_0}{2(2n+2)},$$

$$4(2n+4)a_4 = -a_2, \qquad a_4 = +\frac{a_0}{2.4(2n+2)(2n+4)},$$

$$6(2n+6)a_6 = -a_4, \qquad a_6 = -\frac{a_0}{2.4.6(2n+2)(2n+4)(2n+6)},$$

and so on. The mode of formation is self-evident and the coefficients are uniquely determined so long as n is not a negative integer. We have

$$y = x^{-n}C_n(x) = a_0\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \dots\right].$$

The factor a_0 is arbitrary and it is customary to put

$$a_0 = \frac{1}{2^n\Gamma(n+1)}.$$

On transposing a factor we have

$$(1)\quad J_n(x)=\frac{x^n}{2^n\Gamma(n+1)}\left\{1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2.4(2n+2)(2n+4)}-\dots\right\}.$$

An alternative form is

$$(2)\quad J_n(x)=\frac{(\frac{1}{2}x)^n}{\Gamma(n+1)}\left\{1-\frac{(\frac{1}{2}x)^2}{1(n+1)}+\frac{(\frac{1}{2}x)^4}{1\,.\,2(n+1)(n+2)}-\dots\right\}.$$

This $J_n(x)$ is known as the Bessel Function of the First Kind, of order n. In the case of integral orders the $\Gamma(n+1)$ can be replaced by $n!$. If c is a zero, so that $J_n(c)=0$, then $J_n(-c)$ is obviously zero and no particular interest attaches to negative zeros. We proceed to show that $J_n(x)$ comes in the category of cylinder functions.

The general or $(r+1)$th term can be written

$$\frac{(-)^r(\frac{1}{2}x)^{n+2r}}{\Gamma(r+1)\Gamma(n+r+1)},\qquad r=0,1,2,\dots.$$

Hence the general term of $nJ_n(x)+xJ_n'(x)$ is

$$\frac{(-)^r(\frac{1}{2}x)^{n+2r}}{\Gamma(r+1)\Gamma(n+r+1)}\{n+(n+2r)\}=x\,\frac{(-)^r(\frac{1}{2}x)^{n-1+2r}}{\Gamma(r+1)\Gamma(n+r)}.$$

This is the $(r+1)$th term of $xJ_{n-1}(x)$, so that

$$(3)\qquad nJ_n+xJ_n'=xJ_{n-1},$$

corresponding to the recurrence formula 3·1(3) in the last chapter. The whole of the results proved for cylinder functions can therefore be applied to $J_n(x)$. Thus

$$(4)\qquad nJ_n-xJ_n'=xJ_{n+1},$$

$$(5)\qquad J_{n-1}-J_{n+1}=2J_n',$$

$$(6)\qquad J_{n-1}+J_{n+1}=\frac{2n}{x}J_n,$$

$$(7)\qquad \frac{d}{dx}\left\{x^nJ_n(x)\right\}=x^nJ_{n-1}(x),\quad \frac{d}{dx}\left\{x^{-n}J_n(x)\right\}=-x^{-n}J_{n+1}(x),$$

$$(8)\qquad 2^r\,\frac{d^r}{d(x^2)^r}\left\{x^nJ_n(x)\right\}=x^{n-r}J_{n-r}(x),$$

$$(9)\qquad J_{-1}=J_0'=-J_1,$$

and so on. In particular, if $n = \frac{1}{2}$ we have

$$(10)\quad J_{\frac{1}{2}}(x) = \frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}\Gamma(1\frac{1}{2})}\left\{1-\frac{x^2}{2.3}+\frac{x^4}{2.4.3.5}-\dots\right\} = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\sin x.$$

Since Bessel's equation is unaltered if $-n$ replaces n, we conclude that $J_{-n}(x)$ is equally a solution. The factor outside the bracket is enough to show that it is not a mere numerical multiple of $J_n(x)$; it is accordingly an independent solution and the general solution is

$$y = AJ_n(x) + BJ_{-n}(x).$$

This breaks down when n is an integer, for J_n and J_{-n} are not then linearly independent; in fact we now know, in accordance with cylinder functions, that $J_{-n} = (-)^n J_n$. This result is by no means evident from the series obtained above and the explanation is as follows. Formally we have

$$J_{-n}(x)$$
$$= \frac{(\frac{1}{2}x)^{-n}}{\Gamma(-n+1)}\left\{1-\frac{(\frac{1}{2}x)^2}{1(-n+1)}+\frac{(\frac{1}{2}x)^4}{1.2(-n+1)(-n+2)}-\dots\right\}.$$

Just as every denominator after the second term contains the factor $(-n+2)$, so every denominator after the rth term contains the factor $(-n+r)$; and ultimately, every denominator after the nth term contains the factor $(-n+n)$. This tendency for the terms to become infinite is nullified by the external factor $\Gamma(-n+1)$ which is itself infinite. Accordingly, the first n terms are annihilated and the reader can convince himself that the rest of the series behaves like $J_n(x)$. The troublesome question of the second independent solution for integral orders will be taken up later in the book.

4·2. Behaviour at the origin.

We are now in a position to examine the nature of the function at the origin. Evidently from the series we have $J_n(0) = 0$ for all positive orders; and the same is true for negative integral orders since $J_{-n} = (-)^n J_n$. For all negative non-integral orders, $J_{-n}(0)$ tends to infinity, the sign being that of $\Gamma(-n+1)$, i.e. negative if the greatest integer in n is odd; otherwise positive. This frequently bars the use of the function of negative non-integral order in physical problems of the type where something has to be finite at the origin. The function of zero order is unique; for since

$$(1)\quad J_0(x) = 1 - (\tfrac{1}{2}x)^2 + \frac{(\frac{1}{2}x)^4}{(2!)^2}\dots,$$

we have $J_0(0) = 1$. Hence the function $J_n(x)$ either passes through the origin or goes to infinity, with the single exception of the zero order, which crosses the axis at a height unity (fig. 5).

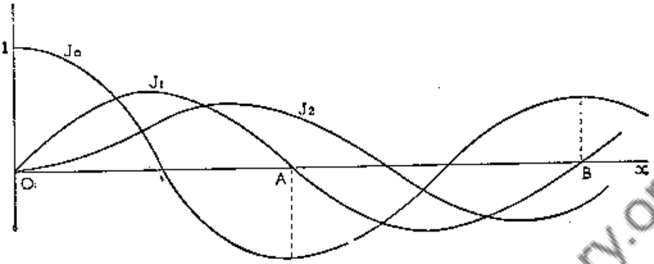

Fig. 5.—March of J_0, J_1 and J_2 near the origin
J_0' and J_1 simultaneously zero at A and B

4·3. The zeros.

Knowing that $J_n(x)$ has an infinity of zeros which, for positive orders, start with the origin, we can exemplify some of our previous work. Thus it was proved that if the order is less than $\frac{1}{2}$, the interval between zeros approaches π from below. The following brief table gives the approximate values of the first five roots of $J_0(x)$ and the corresponding differences.

Root	2·4048	5·5201	8·6537	11·7915	14·9309
Difference		3·1153	3·1337	3·1378	3·1394

Alternatively if the order is greater than $\frac{1}{2}$ the interval approaches π from above. The corresponding table for $J_3(x)$ is:

Root	6·3802	9·7610	13·0152	16·2235	19·4094
Difference		3·3809	3·2542	3·2083	3·1860

We have proved that the first positive zero is greater than n; but we can improve on this by showing that the first positive zero steadily increases with the order. For positive n, m put $u = x^{\frac{1}{2}}J_n$, $v = x^{\frac{1}{2}}J_m$, so that as previously by 3·5(1)

$$u'' + u\left(1 + \frac{1 - 4n^2}{4x^2}\right) = 0,$$

$$v'' + v\left(1 + \frac{1 - 4m^2}{4x^2}\right) = 0,$$

$$(u''v - uv'') + \frac{uv}{x^2}(m^2 - n^2) = 0.$$

It is now legitimate to use the origin as the lower limit of integration.

$$\Big[u'v - uv'\Big]_0^a = (n^2 - m^2)\int_0^a \frac{uv}{x^2}\,dx.$$

Assuming that u vanishes at a whilst v is still positive, consideration of sign shows that m is greater than n and the function of higher order reaches its first zero later. The tables give the first zero for the corresponding order as:

Order	0	1	2	3	4	5
Zero	2·405	3·832	5·136	6·380	7·588	8·771

4·4. Relation between the solutions.

If y_1, y_2 are independent solutions of $y'' + y'f(x) + yg(x) = 0$, we have by 2·5(1)

$$y_1'y_2 - y_1y_2' = A\exp\{-\textstyle\int f(x)\,dx\}.$$

In the case of Bessel's equation with $f(x) = x^{-1}$ the relation becomes

$$J_nJ_{-n}' - J_n'J_{-n} = \frac{A}{x}.$$

We can determine the constant from the series for the four functions involved. The form of the answer being known, only the leading terms need be considered. This gives

$$A = \frac{1}{2^n\Gamma(n+1)}\cdot\frac{-n}{2^{-n}\Gamma(-n+1)} - \frac{n}{2^n\Gamma(n+1)}\cdot\frac{1}{2^{-n}\Gamma(-n+1)}.$$

Since $\Gamma(n+1) = n\Gamma(n)$ this becomes by 1·7(2)

$$\frac{-2}{\Gamma(n)\Gamma(1-n)} = -\frac{2}{\pi}\sin n\pi.$$

Thus

(1) $$J_nJ_{-n}' - J_n'J_{-n} = -\frac{2}{\pi x}\sin n\pi.$$

4·5. The orthogonal property.

Let $u = J_n(\alpha x)$, $v = J_n(\beta x)$ so that by 3·6(1)

$$u'' + \frac{u'}{x} + \left(\alpha^2 - \frac{n^2}{x^2}\right)u = 0,$$

$$v'' + \frac{v'}{x} + \left(\beta^2 - \frac{n^2}{x^2}\right)v = 0.$$

An integration after eliminating n gives

$$(\alpha^2 - \beta^2)\int xuv\,dx = x(uv' - u'v).$$

The right side certainly vanishes at the lower limit zero, provided $2n + 1 > -1$, a stipulation which ensures the integral being convergent. It will vanish at an upper limit of unity if this makes both u and v vanish, or

$$J_n(\alpha) = 0 = J_n(\beta).$$

This means that α and β are two zeros of $J_n(x)$. Replacing them by c_r, c_s (two of the zeros in question) we have

$$(1) \qquad \int_0^1 xJ_n(c_r x)J_n(c_s x)\,dx = 0.$$

This may be expressed by saying that the family of functions $x^{\frac{1}{2}}J_n(c_s x)$ is orthogonal when the c_s are the zeros of $J_n(x)$. The property is useful in expansions, just as the corresponding property for trigonometrical functions is useful in Fourier series.

4·6. Lommel integrals.

It is convenient to treat here certain integrals involving Bessel functions and associated with the name of E. C. J. von Lommel. Using the notation of the previous paragraph, but discarding the assumption that α, β are zeros of $J_n(x)$, we have

$$(1) \quad (\alpha^2 - \beta^2)\int_0^x xJ_n(\alpha x)J_n(\beta x)\,dx$$
$$= x\{\beta J_n(\alpha x)J_n'(\beta x) - \alpha J_n'(\alpha x)J_n(\beta x)\},$$

where the dashes denote differentiation with respect to the argument and not with respect to x. The right side can evidently be thrown into divers forms in virtue of the recurrence formulæ.

As a particular case we have

$$(2) \quad (\alpha^2 - \beta^2)\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = \beta J_n(\alpha)J_n'(\beta) - \alpha J_n'(\alpha)J_n(\beta).$$

An immediate application is to show that $J_n(x)$ cannot have a complex zero. For if $p + iq$ is a complex zero, call it α; and let β denote the conjugate complex $p - iq$, which must also be a zero. The right side of the last equation is then zero by hypothesis; but the integrand contains the product of two conjugate complex functions, which must necessarily be positive. The contradiction establishes that $J_n(x)$ has no complex zero.

A companion result can be obtained by multiplying the equation for u by $2x^2u'$. Thus

$$2x^2u'u'' + 2xu'^2 + 2\alpha^2x^2uu' - 2n^2uu' = 0.$$

Since

$$\frac{d}{dx}(x^2u'^2) = 2x^2u'u'' + 2xu'^2,$$

integration by parts gives

$$x^2u'^2 + \alpha^2x^2u^2 - n^2u^2 = 2\alpha^2\int xu^2\,dx,$$

whence

(3) $$\int xJ_n^2(\alpha x)\,dx = \tfrac{1}{2}x^2\left\{\left(1 - \frac{n^2}{\alpha^2x^2}\right)J_n^2(\alpha x) + J_n'^2(\alpha x)\right\}.$$

In parallel with the foregoing we have the case where the arguments are the same but the orders different. Let $u = J_n(\alpha x)$, $v = J_m(\alpha x)$ so that

$$u'' + \frac{1}{x}u' + \left(\alpha^2 - \frac{n^2}{x^2}\right)u = 0,$$

$$v'' + \frac{1}{x}v' + \left(\alpha^2 - \frac{m^2}{x^2}\right)v = 0.$$

From these we derive

$$x(u''v - uv'') + (u'v - uv') + (m^2 - n^2)\frac{uv}{x} = 0,$$

whence

$$(m^2 - n^2)\int \frac{uv}{x}\,dx = x(uv' - u'v),$$

or,

(4) $$(m^2 - n^2)\int \frac{1}{x}J_n(\alpha x)J_m(\alpha x)\,dx = \alpha x\{J_n(\alpha x)J_m'(\alpha x) - J_n'(\alpha x)J_m(\alpha x)\}.$$

Here again the recurrence formulæ are capable of producing new forms.

Another fruitful method of integrating products, also due to Lommel, may be illustrated by examples. Bearing in mind the formula 4·1(7),

$$\frac{d}{dx}\{x^{-n}J_n\} = -x^{-n}J_{n+1},$$

we differentiate the identity

$$x^rJ_1J_3 \equiv x^{r+4}(x^{-1}J_1)(x^{-3}J_3)$$

and derive

$$\frac{d}{dx}(x^rJ_1J_3) = (r+4)x^{r-1}J_1J_3 - x^r(J_2J_3 + J_1J_4).$$

Similarly, using the companion formula on the identity

$$x^rJ_2J_4 \equiv x^{r-6}(x^2J_2)(x^4J_4)$$

we derive

$$\frac{d}{dx}(x^rJ_2J_4) = (r-6)x^{r-1}J_2J_4 + x^r(J_2J_3 + J_1J_4).$$

By addition,

$$\frac{d}{dx}\{x^r(J_1J_3 + J_2J_4)\} = x^{r-1}\{(r+4)J_1J_3 + (r-6)J_2J_4\}.$$

Since r is at our choice, $r = 6$ gives

$$10\int x^5J_1J_3\,dx = x^6(J_1J_3 + J_2J_4),$$

whilst $r = -4$ gives

$$10\int x^{-5}J_2J_4\,dx = -x^{-4}(J_1J_3 + J_2J_4).$$

As an example where the orders are equal, consider

$$x^rJ_2^2 \equiv x^{r+4}(x^{-2}J_2)^2,$$

whence

$$\frac{d}{dx}\{x^rJ_2^2\} = (r+4)x^{r-1}J_2^2 - 2x^rJ_2J_3.$$

Similarly

$$x^rJ_3^2 \equiv x^{r-6}(x^3J_3)^2,$$

whence

$$\frac{d}{dx}\{x^rJ_3^2\} = (r-6)x^{r-1}J_3^2 + 2x^rJ_2J_3.$$

By addition,

$$\frac{d}{dx}\{x^r(J_2^2 + J_3^2)\} = x^{r-1}\{(r+4)J_2^2 + (r-6)J_3^2\}.$$

Hence

$$10\int x^5J_2^2\,dx = x^6(J_2^2 + J_3^2),$$

$$10\int x^{-5}J_3^2\,dx = -x^{-4}(J_2^2 + J_3^2).$$

EXERCISES

1. Prove that the indicial equation for $x^2y'' + xy' + (x^2 - n^2)y = 0$ is $r^2 = n^2$ and deduce the series for $J_n(x)$.

2. Verify that the second root of the indicial equation in the text, $r = -2n$, leads to the series for $J_{-n}(x)$.

3. Verify that the expansion for $J_n(x)$ is convergent for all values of x. Prove that it is absolutely convergent.

4. Prove that the solution of Bessel's equation is $AJ_n + BJ_n \int \frac{dx}{xJ_n^2}$.

5. Prove from the series that J_n can have no purely imaginary zero.

6. Prove that $J_n'(x)$ has no complex zero. Prove further that $aJ_n + b\,xJ_n'$ cannot have a complex zero. [Assume that α, β are conjugate complex zeros and eliminate a, b.]

7. $J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos x, \quad J_{3/2}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \left(\frac{\sin x}{x} - \cos x\right),$

$J_{-3/2}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \left(-\sin x - \frac{\cos x}{x}\right)$ and evaluate for the orders $2\frac{1}{2}$, $-2\frac{1}{2}$.

8. $J_n J_{-n+1} + J_{-n} J_{n-1} = 2\frac{\sin n\pi}{\pi x},$

$J_n J_{-n-1} + J_{-n} J_{n+1} = -2\frac{\sin n\pi}{\pi x}.$

9. If n is a positive integer,

$$x^{-n-\frac{1}{2}} J_{n+\frac{1}{2}}(x) = (-2)^n \sqrt{\left(\frac{2}{\pi}\right)} \frac{d^n}{d(x^2)^n} \left(\frac{\sin x}{x}\right),$$

$$x^{-n-\frac{1}{2}} J_{-n-\frac{1}{2}}(x) = 2^n \sqrt{\left(\frac{2}{\pi}\right)} \frac{d^n}{d(x^2)^n} \left(\frac{\cos x}{x}\right).$$

10. $\frac{d}{dx}\{x^{\frac{1}{2}n} J_n(x^{\frac{1}{2}})\} = \frac{1}{2} x^{\frac{1}{2}(n-1)} J_{n-1}(x^{\frac{1}{2}}).$

$\frac{d}{dx}\{x^{n\alpha} J_n(kx^\alpha)\} = k\alpha x^{(n+1)\alpha-1} J_{n-1}(kx^\alpha).$

$\frac{d}{dx}\{x^{-n\alpha} J_n(kx^\alpha)\} = -k\alpha x^{(1-n)\alpha-1} J_{n+1}(kx^\alpha).$

11. $\frac{1}{2}xJ_0 = J_1 - 3J_3 + 5J_5 - \ldots.$

12. $\int \frac{dx}{xJ_n^2} = -\frac{\pi}{2\sin n\pi} \frac{J_{-n}}{J_n}.$

$\int \frac{dx}{xJ_nJ_{-n}} = -\frac{\pi}{2\sin n\pi} \log \frac{J_{-n}}{J_n}.$

13. $\int_0^a xJ_0^2(kx)\,dx = \frac{1}{2}a^2\{J_0^2(ka) + J_1^2(ka)\}.$

14. $\int x^{2n+1} J_n{}^2(x)\,dx = \frac{x^{2n+2}}{4n+2}\{J_n{}^2(x) + J^2{}_{n+1}(x)\}.$

$\int x^{-2n-1} J_{n+1}{}^2(x)\,dx = -\frac{x^{-2n}}{4n+2}\{J_n{}^2(x) + J_{n+1}{}^2(x)\}.$

15. $(\beta^2 - \alpha^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{\alpha\beta}{2n}\{J_{n-1}(\alpha)J_{n+1}(\beta) - J_{n+1}(\alpha)J_{n-1}(\beta)\}.$

The integral is zero if α, β are zeros of $aJ_n(t) + btJ_n'(t)$; and when else?

16. $\int x J_n{}^2(\alpha x)\,dx = \frac{1}{2}x^2\{J_n{}^2(\alpha x) - J_{n-1}(\alpha x)J_{n+1}(\alpha x)\}.$

17. If $m + n > 0$, prove that

$$\int_0^1 \frac{1}{x} J_m(\alpha x) J_n(\alpha x)\,dx = \frac{\alpha}{m^2 - n^2}\{J_n(\alpha)J_m'(\alpha) - J_n'(\alpha)J'_m(\alpha)\}.$$

Deduce by a limiting process that

$$\int_0^1 \frac{1}{x} J_n{}^2(\alpha x)\,dx = \frac{\alpha}{2n} J_n{}^2(\alpha)\frac{\partial}{\partial n}\left\{\frac{J_n'(\alpha)}{J_n(\alpha)}\right\}.$$

18. By means of the formula for $d(x^n J_n)/dx$ and its counterpart, prove that

$$\int x^6 J_2(x)\,dx = 2x^5 J_3(x) + x^4(x^2 - 16)J_4(x).$$

19. Establish the reduction formula

$$\int x^{m+1} J_n\,dx + (m^2 - n^2)\int x^{m-1} J_n\,dx = x^{m+1} J_{n+1} + (m - n)x^m J_n.$$

20. Prove that the solution of $xy'' + \frac{1}{2}y' + \frac{1}{4}y = 0$ is

$x^{\frac{1}{4}}\{AJ_{\frac{1}{2}}(\sqrt{x}) + BJ_{-\frac{1}{2}}(\sqrt{x})\}.$

21. Find the general solution of the equation

$x^2y'' - 2xy' + 4(x^4 - 1)y = 0.$

22. If c is a zero of J_n, prove

$$2\int_0^1 x J_n{}^2(cx)\,dx = J_{n+1}{}^2(c) = J_{n-1}{}^2(c) = J_n'^2(c).$$

Deduce that J_n has no zero in common with J_{n+1}, J_{n-1}, J_n' or $aJ_n + bxJ_n'$, except the origin.

23. If c_1, c_2, &c., denote the zeros of J_0, whilst d_1, d_2, &c., denote the zeros of J_1, the origin excepted, prove that the interval $d_r - c_r$ steadily increases with r, and that the interval $c_{r+1} - d_r$ steadily decreases. A partial verification is afforded by the tables; the first five zeros of J_0 are given in the text and the corresponding first five zeros of J_1 are

3·8317 7·0155 10·1734 13·3237 16·4706.

The verification is left to the reader. Extend the theorem to other orders and verify by the roots of J_3 given in the text. Extend the theorem to the cylinder functions. An application is given later in Chap. VI.

24. If c_1, c_2, . . . are the successive zeros of $J_n(x)$ when n is greater than $\frac{1}{2}$, prove that the interval $c_{r+1} - c_r$ decreases with r increasing. Verify from any roots given in the text and extend the result to cylinder functions. [Consider the normal form of the equation satisfied by $x^{\frac{1}{2}}J_n(x)$ and prove that the coefficient of y is an increasing function of x.]

CHAPTER V

Applications

5·1. The lengthening pendulum.

A problem in mathematical physics usually falls into two well-defined parts. In the first place there is some principle which holds throughout the body of the medium under discussion. It may be that the excess of heat-entry over heat-exit for an element is accounted for by a rise of temperature; or it may be that the acceleration is the ratio of the effective force to the mass. Whatever it be, the mathematical expression of such a principle almost invariably takes the form of a differential equation.

In the second place there are special conditions at the outskirts and these are known as end-conditions, or more often, boundary conditions. A rod may be fixed at one end and so be immune from displacement; and a moving fluid, in default of creating vacua, can have no velocity normal to the walls of its container; and so on.

The essential problem then is, so to solve the differential equation that it contains arbitrary constants in number sufficient to satisfy the boundary conditions uniquely. Theory then usually shows that there can be no solution other than this. We propose to investigate a number of such problems which depend for their solution on Bessel functions of the first kind. We begin with a relatively simple dynamical problem that requires but little analysis.

Problem 1.—*Discuss the small oscillations of a simple pendulum when the string is paid out from the support at a constant rate.*

At any time t let the string make angle θ with the downward vertical and let its length be $r = a + bt$. The string is thus paid out at a constant rate $\dot{r} = b$ and the initial length when $t = 0$ is $r = a$. The transverse acceleration of the mass is $r\ddot{\theta} + 2\dot{r}\dot{\theta}$, perpendicular to the string in the direction of θ increasing. This acceleration is retarded by the component of the weight, the equation of motion being

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -mg \sin\theta.$$

For small oscillations we replace $\sin\theta$ by θ, and on substituting for r and $\dot{r}$ we have

$$(a + bt)\ddot{\theta} + 2b\dot{\theta} + g\theta = 0,$$

a reduced linear differential equation of the second order with variable coefficients. If the leading coefficient be made unity, there is a single value of t for which the other coefficients become infinite. This suggests that the equation may be of Bessel's type; but as the infinity occurs when $t = -a/b$ and not when $t = 0$ we change the origin of time by the substitution $a + bt = bx$, $dt = dx$. We then have

$$(1) \qquad \frac{d^2\theta}{dx^2} + \frac{2}{x}\frac{d\theta}{dx} + \frac{k^2}{x}\theta = 0; \quad k^2 = \frac{g}{b}.$$

This is not directly comparable with Bessel's equation and one can spend much time seeking the necessary transformation that converts the one into the other. The readiest method is to compare with the general form 3·6(2). We thus deduce

$$1 - 2\alpha = 2, \quad (\beta\gamma x^{\gamma-1})^2 = \frac{k^2}{x}, \quad \alpha^2 = n^2\gamma^2,$$

$$\alpha = -\tfrac{1}{2}, \qquad \gamma = \tfrac{1}{2}, \qquad n = \pm 1,$$

$$\beta = 2k.$$

Since J_{-1} is not different from J_1 except for sign, we can take as a limited solution

$$\theta = Ax^{-\frac{1}{2}}J_1(2kx^{\frac{1}{2}}).$$

The constant A remains arbitrary since the oscillations are merely described as small without particularizing. We can find conditions under which this solution is valid by going to the initial conditions. With $t = 0$ we have $r = a$, $x = a/b$, so that the initial value of the angle must have been

$$\theta_0 = A\sqrt{\left(\frac{b}{a}\right)}J_1\left\{\frac{2}{b}\sqrt{(ag)}\right\}.$$

It is not possible to say whether θ is increasing or decreasing until we know more about the constants. At a later stage, when we have discussed the second solution for integral orders, we shall be able to discuss the problem with ampler initial conditions of length, angular displacement and velocity. In the meantime we append a few exercises.

EXERCISES

1. Verify that the angular velocity and the angular acceleration are

$$-\frac{Ak}{x}J_2(2kx^{\frac{1}{2}}), \qquad +\frac{Ak^2}{x^{3/2}}J_3(2kx^{\frac{1}{2}}),$$

and compare the remark on this form in 3·6(4).

2. It is dynamically evident that between two consecutive positions of temporary angular rest is a single instant of zero displacement; and conversely. Interpret this in terms of the Bessel functions. (There is no temporary absolute rest since $\dot{r} = b$.)

3. It is known that the graphs of the three functions J_1, J_2, J_3 cross the x-axis, from above or below, in that order. It is further known that the zeros of any pair of the functions interlace (see 3·6, Ex. 9). Moreover, from the equation of motion, if any one of $\theta, \dot{\theta}, \ddot{\theta}$ be zero the other two have opposite sign. Prove by the aid of a rough sketch that these statements corroborate one another.

4. In the case of the ordinary simple pendulum it is known that, (i) whether the angular displacement be positive or negative, the angular acceleration is always directed to the equilibrium position; and (ii) the displacement and acceleration vanish simultaneously. In the present case neither statement holds. What is the interpretation in terms of the Bessel functions?

5. Prove that the angular acceleration changes sign only once in each swing, and that it happens before the vertical position is reached.

6. Temporary rest is equivalent to maximum displacement; and zero acceleration corresponds to maximum velocity. Interpret these.

7. Eliminate the time from the differential equation of motion and so find a solution for θ in terms of r.

8. Prove that the time-interval between two consecutive transits through the vertical is $b(c_{r+1}{}^2 - c_r{}^2)/4g$, where c_r, c_{r+1} are consecutive zeros of $J_1(t)$. The interval therefore increases indefinitely.

9. From the principle of angular momentum, that the rate of change of angular momentum equals the moment of the impressed forces, we have

$$\frac{d}{dt}\left\{mr^2\frac{d\theta}{dt}\right\} = -mgr\theta,$$

whence

$$r^2\frac{d\theta}{dt} = -g\int r\theta\,dt.$$

Deduce that

$$xJ_2(2kx^{\frac{1}{2}}) = k\int x^{\frac{1}{2}}J_1(2kx^{\frac{1}{2}})\,dx.$$

10. It is interesting to speculate on how what one might call the "quarter periods" compare with each other. Continuing the notation of Ex. 8, let d_r, d_{r+1}

be corresponding consecutive zeros of $J_2(t)$. The quarter period for an outward swing from the vertical is proportional to $d_r^2 - c_r^2$. The ensuing quarter period for the inward swing to the vertical is proportional to $c_{r+1}^2 - d_r^2$, and so on. The sixth to tenth zeros of $J_1(t)$ taken from the tables are

19·616   22·760   25·904   29·074   32·190.

The sixth to ninth zeros of $J_2(t)$ are

21·117   24·270   27·421   30·569.

These show that the successive quarter periods, beginning with an outward movement, are proportional to

Out:	60·14	71·02	80·89	90·76.	
In:		72·09	81·96	91·83	101·70.

The curious result emerges that an inward half-swing takes longer than either the preceding or succeeding outward half-swing. I see no physical reason for this; nor do I know of any proof. It would be equivalent to showing that

$$d_r^2 - c_r^2 < c_{r+1}^2 - d_r^2 > d_{r+1}^2 - c_{r+1}^2,$$

or,

$$c_r^2 + c_{r+1}^2 > 2d_r^2; \quad d_r^2 + d_{r+1}^2 < 2c_{r+1}^2.$$

5·2. Vibrations of a taut string.

Consider now the transverse vibrations of a taut string. The natural direction of the string being along the x-axis, consider a displaced element AB of length δs; the co-ordinates of A being x, y, those of B are $x + \delta x$, $y + \delta y$. Let T, making angle ψ with OX, be the tension at A; its components X, Y, being to the left and downwards respectively, so that $X = T\cos\psi$, $Y = T\sin\psi$. The corresponding quantities at B are $X + \delta X$, $Y + \delta Y$, $T + \delta T$. The resultant horizontal force is δX; and as it is customary to assume the longitudinal motion to be negligible, we have δX zero. This gives $X = \text{const.} = T\cos\psi$.

In practice the angle ψ is too small to be seen with the naked eye, so it is a legitimate approximation to replace $\cos\psi$ by unity. This leads to the usual assumption that the tension is constant throughout the length. At right angles to the string the resultant force upwards is $\delta Y = \delta(T\sin\psi)$. If ρ be the line-density, not necessarily constant, the mass of the element is $\rho\,\delta s$; and as its displacement is y, its acceleration is $\ddot{y}$. This gives the equation of motion

$$\frac{\partial^2 y}{\partial t^2}\,\rho\,\delta s = \delta Y = \frac{\partial}{\partial s}(T\sin\psi)\,\delta s.$$

As $\sin\psi = dy/ds$ and the distinction between x and s is not worth making, we have

$$\rho\,\frac{\partial^2 y}{\partial t^2} = T\,\frac{\partial^2 y}{\partial x^2}.$$

So far we have followed the routine procedure. We now propose to modify the problem so as to admit of treatment by Bessel functions.

Problem 2.—*Discuss the transverse vibrations of a non-uniform taut string whose line density at distance x from one end is $\rho(1+kx)$.*

This may be regarded as a first approximation to a wet string in a vertical position. The equation of motion is

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}(1+kx)\,\frac{\partial^2 y}{\partial t^2}.$$

On the ground that any periodic variation of the displacement with time can be expressed in a Fourier series we write

$$y = u\sin(pt+\phi), \quad \frac{\partial^2 y}{\partial t^2} = -p^2 u\sin(pt+\phi).$$

This introduces a number p whose different values determine the frequencies of the various harmonics. The equation of motion is now

$$\frac{d^2u}{dx^2} + \frac{p^2\rho}{T}(1+kx)u = 0.$$

A slight modification is required to achieve a tractable form; we put

$$1+kx = kz, \quad dx = dz,$$

whence

$$\frac{d^2u}{dz^2} + p^2\,\frac{\rho}{T}\,kzu = 0.$$

The equation is in the normal form, so if it turns out to be soluble in terms of Bessel functions they will contain the factor $z^{\frac{1}{2}}$. Comparing as before with the general standard we have by 3·6(2)

$$\alpha = \tfrac{1}{2}, \quad \gamma = \tfrac{3}{2}, \quad n = \pm\tfrac{1}{3},$$

$$\beta^2\gamma^2 = \frac{9\beta^2}{4} = p^2\,\frac{\rho}{T}\,k, \qquad \beta = \tfrac{2}{3}p\sqrt{\left(\frac{\rho k}{T}\right)}.$$

A solution is therefore

$$u = Az^{\frac{1}{2}}J_{\frac{1}{3}}(\beta z^{3/2}) + Bz^{\frac{1}{2}}J_{-\frac{1}{3}}(\beta z^{3/2}).$$

It remains to determine the constants A and B from the fact that both ends are fastened. Taking the origin at the left end of the string, we have as end-conditions

$$x = 0 = y = u, \quad z = \frac{1}{k};$$

$$x = \lambda, \quad y = 0 = u, \quad z = \left(\frac{1}{k} + \lambda\right).$$

These give

$$0 = AJ_{\frac{1}{3}}(\beta k^{-3/2}) + BJ_{-\frac{1}{3}}(\beta k^{-3/2}),$$

and

$$0 = AJ_{\frac{1}{3}}\left\{\beta\left(\frac{1}{k} + \lambda\right)^{3/2}\right\} + BJ_{-\frac{1}{3}}\left\{\beta\left(\frac{1}{k} + \lambda\right)^{3/2}\right\}.$$

Putting for brevity

$$\beta k^{-3/2} = c, \quad (1 + k\lambda)^{3/2} = n,$$

the elimination of A, B gives

$$J_{\frac{1}{3}}(c)J_{-\frac{1}{3}}(nc) = J_{-\frac{1}{3}}(c)J_{\frac{1}{3}}(nc).$$

The number n is known from the conditions of the problem. The various values of c determine the corresponding values of β, and these in turn determine p and the possible frequencies. This is a fair sample of the type of problem one encounters, and all numerical work in connexion with it depends on our ability to solve the transcendental equation for c. Tables of the function of order $\frac{1}{3}$ are available and the solution of a transcendental equation is a matter of approximation, by Newton's or some similar method; but the details of the procedure lie outside our scope.

If we write

$$f(x) = f(nx) = J_{\frac{1}{3}}(x)/J_{-\frac{1}{3}}(x)$$

the graph of $y = f(x)$ has the zeros of $J_{\frac{1}{3}}$. When $J_{-\frac{1}{3}}$ passes through a zero, the graph switches from $+\infty$ to $-\infty$, or conversely. The graph accordingly looks somewhat akin to the graph of $\tan x$. The graph of $y = f(nx)$ is the same except that there is horizontal magnification to scale n. It follows that the equation for c has an infinity of roots, as a rough sketch will show. The ratio A/B is therefore determinate; but one of them remains arbitrary. The most general solution of the partial differential equation for the displacement is

$$y = u_1(z)\sin(p_1t + \phi_1) + u_2(z)\sin(p_2t + \phi_2) + \dots .$$

Such constants as are still present can theoretically be determined to conform with initial conditions of shape and velocity.

It remains to examine the question of nodes. The displacement y, and consequently u, is permanently zero if

$$\frac{J_{\frac{1}{3}}(\beta z^{3/2})}{J_{-\frac{1}{3}}(\beta z^{3/2})} = -\frac{B}{A} = \frac{J_{\frac{1}{3}}(\beta k^{-3/2})}{J_{-\frac{1}{3}}(\beta k^{-3/2})} = \frac{J_{\frac{1}{3}}(c)}{J_{-\frac{1}{3}}(c)}.$$

For a permissible value of c the fraction on the right is a definite number. The function on the left, as already explained, ranges from $+\infty$ to $-\infty$ and therefore takes all intermediate values repeatedly, so that there will be nodes when the vibration exceeds the gravest node in frequency.

EXERCISES

1. Taking the line density as $\rho(1 + kx)^2$, prove that the solution depends on

$$z^{\frac{1}{2}}J_{\frac{1}{4}}(\beta z^2), \quad \beta = \tfrac{1}{2}pk\sqrt{\left(\frac{\rho}{T}\right)}$$

with the corresponding function of negative order. Deduce the equation which determines the possible frequencies.

2. Investigate the problem when the line density is $\rho(1 + kx)^{\frac{1}{2}}$, showing that the solution depends on functions of order $\frac{2}{5}$, $-\frac{2}{5}$.

3. When the line density is $\rho(1 + kx)^{-\frac{1}{2}}$ the necessary functions are of order $\frac{2}{3}$, $-\frac{2}{3}$.

4. Solve the problem when the density is proportional to x^a and the ends are defined by $x = \lambda_1, \lambda_2$.

5. Prove that the case where the law is $\rho(1 + kx)^{-2}$ is not soluble in terms of Bessel functions. Also that the law $\rho(1 + kx)^{-1}$ requires two functions of the first order.

6. Prove that a law of density can always be found so that a string of given length can vibrate in the shape (apart from scale) of any specified loop or series of loops of a function $x^{\frac{1}{2}}J_n(\beta x)$, where n is assigned and β remains to be found.

5·3. Stability of a vertical wire.

If a straight wire be clamped vertically at its lower end, the position is stable when the wire is short. A longer wire may find a more stable equilibrium in a curved position. We propose to investigate this.

Problem 3.—A uniform wire of length λ and weight w per unit length has its lower end clamped vertical. Discuss the condition of instability.

Measuring from the lower end O, consider two adjacent points A, B such that $OA = x$, $AB = \delta x$. Presuming that A is situated a distance y from the vertical through O, the corresponding distance for B is $y + \delta y$. At the onset of instability, when y is small, the bending moment M at A is given with sufficient accuracy by $M = EIy''$, where E is Young's modulus for the material and I is the moment of inertia of the cross-section. The weight of wire above B is not sensibly different from that above A; but it acts at a shorter leverage. The bending moment $M + \delta M$ at B is accordingly slightly less than that at A and we have

$$-\delta M = w(\lambda - x)\delta y = -\frac{dM}{dx}\delta x.$$

Put

$$\frac{dy}{dx} = p, \quad M = EI\frac{dp}{dx},$$

then

$$\frac{dM}{dx} = EI\frac{d^2p}{dx^2} = -w(\lambda - x)p.$$

The form of this equation suggests to an experienced eye that any series solution is likely to be in powers of $(x - \lambda)$ rather than x. We accordingly put

$$\lambda - x = z, \quad dx = -dz, \quad \frac{w}{EI} = \frac{9}{4}k^2.$$

The equation becomes

$$\frac{d^2p}{dz^2} + \frac{9zk^2}{4}p = 0.$$

It is in the normal form and therefore if it turns out to be soluble in terms of Bessel functions, they will contain the factor $z^{\frac{1}{2}}$. Comparison with the general form 3·6(2) gives

$$\alpha = \tfrac{1}{2}, \quad \gamma = \tfrac{3}{2}, \quad n = \pm\tfrac{1}{3}, \quad \beta = k.$$

The solution is therefore

$$p = Az^{\frac{1}{2}}J_{\frac{1}{3}}(kz^{3/2}) + Bz^{\frac{1}{2}}J_{-\frac{1}{3}}(kz^{3/2}).$$

It remains to determine the constants.

At the upper end, $x = \lambda$, $z = 0$ and as there is no applied couple we have

$$y'' = 0 = \frac{dp}{dx} = -\frac{dp}{dz}.$$

In view of previous experience it is possible to write down (4·6, Ex. 10)

$$\frac{dp}{dz} = \tfrac{3}{2}AkzJ_{-\frac{2}{3}}(kz^{3/2}) - \tfrac{3}{2}BkzJ_{\frac{2}{3}}(kz^{3/2}).$$

It is left to the reader to verify that the series for the former term starts with a constant, whereas that for the second terms starts with z^2. Since dp/dz vanishes at $z = 0$, the value of the constant must be zero, whence A is zero. At the foot of the wire, on account of the clamping, we have

$$x = 0, \quad z = \lambda, \quad p = 0.$$

These give

$$0 = B\lambda^{\frac{1}{2}}J_{-\frac{1}{3}}(k\lambda^{3/2}).$$

The only rational way of satisfying this equation is to take the argument $k\lambda^{3/2}$ as the first zero of $J_{-\frac{1}{3}}(x)$. This is known to be approximately 1·87. In any given case where w, E and I are known, k is determined and hence λ. It remains to add that if the wire is a ribbon or of any cross-section other than circular, the instability occurs in the direction of greatest flexibility and the appropriate value of I must be chosen.

"I have no satisfaction in formulas," said Lord Kelvin, "unless I feel their arithmetical magnitude—at all events when formulas are intended for definite dynamical or physical problems." For a wire of radius r made of material of density ρ we have

$$w = \pi r^2 g\rho, \quad I = \tfrac{1}{4}\pi r^4,$$

$$\frac{w}{EI} = \tfrac{9}{4}k^2 = \frac{4g\rho}{Er^2}, \qquad k = \frac{4}{3r}\sqrt{\left(\frac{g\rho}{E}\right)}.$$

A reasonable value for E is 2.10^3 tonnes/cm.^{2}, or $2g \,.\, 10^9$ dynes/cm.^{2}. If the diameter is 1 mm. and the density is 7·6 gr./c.c., this gives $k = 1{\cdot}64 \times 10^{-3}$ and $\lambda = 109$ cm.

EXERCISES

1. It states in the text that the only rational way of satisfying the condition of stability is to take the smallest zero of $J_{-\frac{1}{3}}(x)$. What is the objection, if any, to choosing zeros other than the smallest?

2. Adopting the solution for p given in the text, with the accompanying value of dp/dz, verify that the normal form is satisfied.

3. If the wire carries a top weight W, prove that the greatest length for stability depends on the first root of the equation

$$J_{\frac{1}{3}}(n\alpha)J_{\frac{2}{3}}(\alpha) + J_{-\frac{1}{3}}(n\alpha)J_{-\frac{2}{3}}(\alpha) = 0;$$

$$\alpha = k\left(\frac{W}{w}\right)^{3/2}, \quad n = \left(1 + \frac{\lambda w}{W}\right)^{3/2}, \quad 9k^2 = \frac{4w}{EI}.$$

4. Since

$$p = \frac{dy}{dx} = -\frac{dy}{dz},$$

we have $y = -\int p\, dz$ to determine the form of the wire. Prove by a change of variable that this is equivalent to $\int J_{-\frac{1}{3}}(t)\, dt$. The integral cannot be evaluated in finite terms.

5. If the wire is a solid of revolution, the radius r at depth x below the top being λx^m and the weight above the section being μx^n, show that the solution depends on $J_p(x) = 0$, where $p = \dfrac{4m-1}{n-4m+2}$.

5·4. Instability of the deep cantilever.

It is well known that a deep cantilever tends to lateral instability, especially if the load rides high. The reader can easily convince himself of this by using a horizontal strip of paper carrying a paper-clip at the end of its upper edge. We propose to investigate this, taking the case of end-load on the centre line.

Problem 4.—*Discuss the instability of the deep cantilever under end-load.*

For convenience we take the x axis to the right, y axis vertically upwards, the z axis towards the reader, and gravity acting upwards (fig. 6). The length of the beam being λ and end-load W, the centre-line becomes a tortuous curve which was originally the x axis. Any point B on it has co-ordinates x, y, z and its end-point is with sufficient accuracy λ, y_0, z_0. The cross-section at B is presumed to have turned through angle θ clockwise when viewed from the origin.

There are two component couples at B which we may denote by C_x, C_z. Their magnitudes and direction cosines are

$$C_x = W(z - z_0); \qquad 1,\ 0,\ 0.$$
$$C_z = W(\lambda - x); \qquad 0,\ 0,\ 1.$$

It is presumed that a plane section remains plane and normal to the centre line; the normal to the section at B thus has direction cosines

$$\frac{dx}{ds},\ \frac{dy}{ds},\ \frac{dz}{ds},$$

where ds is a line-element of the centre line.

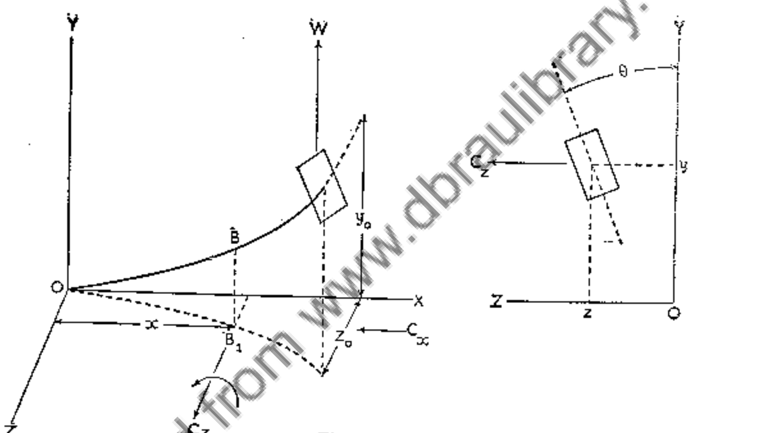

Fig. 6

As the twist per unit length is proportional to the normal torque, we have

$$C_z \frac{dz}{ds} + C_x \frac{dx}{ds} = A \frac{d\theta}{ds},$$

where A is a constant of proportionality depending on the cross-section and the material. We can differentiate this with respect to s, whence

$$\frac{d}{ds}\left\{C_z \frac{dz}{ds} + C_x \frac{dx}{ds}\right\} = A \frac{d^2\theta}{ds^2}.$$

On the grounds that x and s are scarcely distinguishable we put

$$\frac{dx}{ds} = 1, \quad \frac{dC_z}{ds} = -W, \quad \frac{dC_x}{ds} = W \frac{dz}{ds},$$

whence
$$C_z \frac{d^2z}{ds^2} = A \frac{d^2\theta}{ds^2} = W(\lambda - x) \frac{d^2z}{ds^2}.$$

A line which was originally vertical on the section at B has direction cosines 0, $\cos\theta$, $\sin\theta$; or, 0, 1, θ approximately, on the grounds that θ is small. If the couple in this direction be offset against the corresponding bending we have

$$C_z\theta = W(\lambda - x)\theta = -EI\frac{d^2z}{ds^2}.$$

Eliminating z from these two equations we reach

$$EIA\frac{d^2\theta}{dx^2} + W^2(\lambda - x)^2\theta = 0.$$

This is soluble in terms of Bessel functions if we make some minor changes. Put

$$\lambda - x = t, \quad dx = -dt, \quad \frac{W^2}{EIA} = k^2,$$

whence

$$\frac{d^2\theta}{dt^2} + k^2t^2\theta = 0,$$

which is in the normal form. Comparison with our standard gives by 3·6(2)

$$\alpha = \tfrac{1}{2}, \quad \gamma = 2, \quad \beta = \tfrac{1}{2}k, \quad n = \tfrac{1}{4}, -\tfrac{1}{4}.$$

The solution is therefore

$$\theta = Pt^{\frac12}J_{\frac14}(\tfrac{1}{2}kt^2) + Qt^{\frac12}J_{-\frac14}(\tfrac{1}{2}kt^2).$$

The arbitrary constants P, Q remain to be determined. At the free end $x = \lambda$, $t = 0$ we have no applied couple and

$$\frac{d\theta}{dx} = 0 = \frac{d\theta}{dt}.$$

On referring again to 4·6 Ex. 10 we can write down

$$\frac{d\theta}{dt} = Pkt^{3/2}J_{-\frac34}(\tfrac{1}{2}kt^2) - Qkt^{3/2}J_{\frac34}(\tfrac{1}{2}kt^2).$$

It is easy to verify that the series for the former term begins with a constant, whence we conclude that P is zero.

At the clamped end we have

$$x = 0, \quad t = \lambda, \quad \theta = 0 = Q\lambda^{\frac12}J_{-\frac14}(\tfrac{1}{2}k\lambda^2).$$

Tables give the first zero of $J_{-\frac14}(x)$ as 2·0063, so that approximately

$$\lambda = 2k^{-\frac12}, \quad \text{or} \quad 2/\sqrt{k}.$$

An equivalent to the foregoing analysis is usually accepted in texts on structures or materials; in practice, the tendency to instability is obviated by modifying the section of the beam. An alternative discussion based on a different set of ideas will be found in Temple and Bickley, *Rayleigh's Principle* (Oxford).

EXERCISES

1. Prove that if the load is uniformly distributed the solution depends on functions of order $1/6, -1/6$. The smallest zero is given by $J_n(x) = 0$, $n = -1/6$, $x = 2{\cdot}1423$.

2. If the load varies uniformly from zero at the free end, the solution depends on functions of order $\pm 1/8$. The smallest zero is given by $J_n(x) = 0$, $n = -1/8$, $x = 2{\cdot}209$.

3. Discuss the stability of the centrally loaded deep beam when both ends are encastrée. Prove that it is not fundamentally different from the case discussed in the text.

5·5. Critical load for a variable strut.

One of the simplest applications of the theory of linear differential equations with constant coefficients is to the theory of struts. It is presumed that the reader has some acquaintance with this; it can be found in numerous texts on structures or materials.

The problem in its simplest form is best visualized as the equilibrium position of a flat steel ribbon whose ends are joined by an inextensible string of natural length slightly less than that of the ribbon. Taking the line of the string as x axis and one end as origin, let y measure the departure from the straight at any point. The couples give as the approximate equation of equilibrium $EIy'' + Py = 0$, where P is the tension in the string, E is Young's modulus and I is the moment of inertia of the section. If these be taken as constants, a suitable solution is $y = A \sin nx$ where A is arbitrary and $n^2 = P/EI$.

The deflection is certainly zero at the end where x is zero. Taking λ as the length of the string, the absence of deflection at the other end implies $\sin n\lambda = 0$, whence $n\lambda = \pi, 2\pi, 3\pi, \ldots$. A chosen one of these determines n and thence P. The inference is that a possible equilibrium position is any number of loops of a sine curve. All these are very unstable except the single loop.

Problem 5.—Discuss the stability of a triangular strut.

As a first modification of the analysis sketched above, we assume the ribbon cut to a triangular shape. The moment of inertia I is then proportional to the width at any point, and this in turn is proportional to the distance from the vertex. Taking the length of the ribbon as x-axis and the vertex as the origin, we can replace I by Hx. The equation of equilibrium is then presented to us in the normal form

$$EHxy'' + Py = 0,$$

or,
$$\frac{d^2y}{dx^2} + \frac{k^2}{4x}y = 0, \quad k^2 = \frac{4P}{EH}.$$

The solution must contain the factor $x^{\frac{1}{2}}$, and a comparison with our general standard 3·6(2) gives

$$a = \tfrac{1}{2} = \gamma, \quad n = 1, \quad \beta = k.$$

An acceptable solution is therefore

$$y = Ax^{\frac{1}{2}}J_1(kx^{\frac{1}{2}}).$$

This certainly gives zero deflection at the left where x is zero. For zero deflection at the right where $x = \lambda$ we have $J_1(k\lambda^{\frac{1}{2}}) = 0$, so that $k\lambda^{\frac{1}{2}}$ is a zero of $J_1(x)$. Any particular zero serves to determine k, and this in turn determines P. The smallest zero corresponds to the stable position of a single loop. A configuration in several loops is theoretically possible but would be highly unstable; it corresponds to a higher zero of J_1.

When the ribbon stands in a single loop there is a single point of maximum displacement, given by

$$\frac{dy}{dx} = 0 = \tfrac{1}{2}AkJ_0(kx^{\frac{1}{2}}).$$

This corresponds to the theorem that between two consecutive zeros of J_1 is a single zero of J_0.

Translating into figures, let 25 cm. be the length, 1 mm. the uniform thickness, $2\frac{1}{2}$ cm. the width at the broad end. This gives H calculated at unit distance as $8{\cdot}3\ 10^{-6}$ cm.^{4}. Taking E as $2\,.\,10^9$ gr./cm.^{2} and the first zero of J_1 as 3·8317 we have $k = 0{\cdot}7663$ and $P = 2{\cdot}45$ kg.

Attention is directed to the following point for future reference. The point of maximum displacement is given by $J_0(kx^{\frac{1}{2}}) = 0$ and the first admissible value is $kx^{\frac{1}{2}} = 2{\cdot}4048$. Since $k\lambda^{\frac{1}{2}} = 3{\cdot}8317$ we have $x/\lambda = 0{\cdot}3938$. This is definitely displaced from the middle towards the more flexible end.

EXERCISES

1. If the ribbon is blunt-nosed, with $I = Hx^{\frac{1}{2}}$, prove that the solution is

$$y = Ax^{\frac{1}{4}}J_{\frac{2}{3}}(kx^{\frac{3}{4}}), \quad k = \tfrac{4}{3}\left(\frac{P}{EH}\right)^{\frac{1}{2}}.$$

2. If the ribbon is sharp-nosed, with $I = Hx^{3/2}$, prove that the solution is

$$y = Ax^{\frac{1}{4}}J_2(kx^{\frac{1}{4}}), \quad k = 4\left(\frac{P}{EH}\right)^{\frac{1}{2}}.$$

Deduce that between two consecutive zeros of J_2 is a zero of J_1.

3. Prove that the case $I = Hx^2$ is not soluble in terms of Bessel functions.

4. In any case where the ribbon stands in a single loop, what is the physical justification for saying there are no inflections? Explain this by the equilibrium equation and interpret it in terms of the Bessel functions in any particular case.

5·6. Railway transition curves, Fresnel's integrals.

It is distinctly unusual to find Bessel functions associated with geometry; but the following problem is essentially geometrical. When a locomotive rounds a curve there is a definite side-thrust on the track, and the reaction supplies the necessary centrifugal force. In practice the matter is minimized by giving a super-elevation to the outer rail, thus canting the engine inwards. So long as the running is on a straight track there is no centrifugal force. It follows that if the straight track joins on to a curve of finite radius the force jumps from zero to some definite value as the engine passes the join. This is practically an impulse and is detrimental both to the permanent way and to the comfort of passengers. Moreover, it means reducing speed and losing time for the sake of taking things easy. The problem therefore is so to design the transition curve as to avoid the necessity for reducing speed and to ensure that the centrifugal force rises uniformly from zero.

So long as the engine runs at a constant speed it is immaterial whether the uniformity is with regard to time or space. If P be any point on the curve, let ρ be the radius of curvature and s the arcual distance, measured from some point to be specified later. For a constant speed the centrifugal force is proportional to ρ^{-1}, and if it is to change at a constant rate we have

$$\frac{d\rho^{-1}}{ds} = 2b^2.$$

The constant is given the form $2b^2$ as a matter of convenience. Hence

$$\rho^{-1} = 2b^2 s + c,$$

and the constant c can be eliminated by measuring the arc from the join with the straight track. Since $\rho = ds/d\psi$ we have

$$\frac{d\psi}{ds} = 2b^2 s, \quad \psi = b^2 s^2.$$

No integration constant is required. Taking the origin at the join, and the x-axis as the straight track continued, we have

$$\frac{dx}{ds} = \cos\psi = \cos(bs)^2, \quad \frac{dy}{ds} = \sin\psi = \sin(bs)^2.$$

Hence

$$x = \int_0 \cos(bs)^2\, ds, \quad y = \int_0 \sin(bs)^2\, ds.$$

The change of variable $bs = t$ gives

$$bx = \int_0 \cos t^2\, dt, \quad by = \int_0 \sin t^2\, dt.$$

These integrals make their appearance in a subject about as far removed from the crudities of rolling stock as well could be; they are known as Fresnel's integrals and they occur in the diffraction of light. If we put

$$t^2 = u, \quad dt = \tfrac{1}{2}u^{-\frac{1}{2}}\, du,$$

we have

$$2\left(\frac{2}{\pi}\right)^{\frac{1}{2}} bx = \int_0 \left(\frac{2}{\pi u}\right)^{\frac{1}{2}} \cos u\, du = \int_0 J_{-\frac{1}{2}}(u)\, du.$$

Similarly

$$2\left(\frac{2}{\pi}\right)^{\frac{1}{2}} by = \int_0 \left(\frac{2}{\pi u}\right)^{\frac{1}{2}} \sin u\, du = \int_0 J_{\frac{1}{2}}(u)\, du.$$

They can be evaluated by the repeated use of the recurrence relation

$$2J_n' = J_{n-1} - J_{n+1}.$$

We have in succession

$$2J_{\frac{1}{2}}' = J_{-\frac{1}{2}} - J_{3/2},$$

$$2J_{5/2}' = J_{3/2} - J_{7/2},$$

$$2J_{9/2}' = J_{7/2} - J_{11/2},$$

and so on. Hence by summation, since $J_n(x)$ obviously $\to 0$ when $n \to \infty$,

$$\tfrac{1}{2}J_{-\frac{1}{2}} = J_{\frac{1}{2}}' + J_{5/2}' + J_{9/2}' + \cdots .$$

so that

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} bx = J_{\frac{1}{2}} + J_{5/2} + J_{9/2} + \cdots .$$

The series is convergent and numerical results can be obtained by the use of the tables. The companion result is easily shown to be

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} by = J_{3/2} + J_{7/2} + J_{11/2} + \cdots .$$

They are both apparently due to Lommel. It need hardly be added that no civil engineer could be induced to utilize such results in laying out the curve.

CHAPTER VI

The Second Solution; Further Applications

6·1. The indicial equation.

If the series $y = a_0x^r + a_1x^{r+1} + \ldots$ be substituted in Bessel's equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0,$$

it follows as in 4·1 that the resulting indicial equation is $r^2 = n^2$. The verification is quite simple; for the coefficient of a_0 on substitution is $r(r-1) + r - n^2$. This agrees with the result stated and the roots of the indicial equation are $r = n, -n$. In general the two solutions J_n and J_{-n} are independent since the one is obviously not a mere numerical multiple of the other. Exceptionally, when n is integral, J_{-n} is merely J_n except possibly as regards sign. We now have to investigate the second independent solution for integral orders.

Certain general propositions concerning the indicial equation will be mentioned here. They are;

(i) If the roots of the indicial equation are distinct and do not differ by an integer, each root leads to an independent solution.

(ii) If the roots differ by an integer, the second solution may, or may not, contain a logarithmic term.

(iii) In the case of repeated roots, the second solution certainly contains a logarithmic term.

These are part of the theory of differential equations and no proofs will be attempted here beyond what applies to Bessel's equation, which is the almost perfect exemplar. They are already substantiated to some extent; for the roots of the indicial equation differ by $2n$, and if this is not an integer then neither is n, in which case the two solutions are distinct. If $2n$ is an odd integer, then in general the second solution may, or may not, contain a logarithmic term. In the case of Bessel's equation it does not; for the functions of order half an odd integer can be expressed in terms of $\sin x$ and $\cos x$. When $2n$ is an even integer it will appear later that the second solution of Bessel's equation does contain a logarithmic term. Finally if n is zero the indicial equation has a repeated root and theory states that the second solution must contain a logarithmic term. The suggestion made in 3·6 (for the case $\gamma = 0$) was not devoid of purpose; it can now be disclosed that the solution in question is $x^\alpha \log x$.

6·2. Equations of zero order.

We now proceed to verify the statement about zero order for Bessel's equation; and we begin with a method that makes the result look plausible. From

$$J_0(x) = 1 - \frac{x^2}{4} + \ldots$$

we have by reversion of series

$$\frac{1}{J_0(x)} = 1 + \frac{x^2}{4} + \ldots .$$

Hence
$$\frac{1}{x\{J_0(x)\}^2} = \frac{1}{x} + \frac{x}{2} + \ldots .$$

From 4·6, Ex. 4, and 2·5(2) it is known that the second solution can be expressed as

$$y = J_0(x)\int\frac{dx}{x\{J_0(x)\}^2},$$

whence we deduce that

$$y = J_0(x)\left\{\log x + \frac{x^2}{4} + \ldots\right\},$$

the first term being followed by a series of ascending powers of x.

It would be tedious by this method to find the actual form of the series, which is in fact of no great use to us. The important point is that the value of y is infinite when x is zero, from which it follows that the solution is linearly independent of J_0. It is customary to denote it by Y_0. The presence of $\log x$ confines us to positive values of x, but in practice this is no handicap.

Having given our statement an air of verisimilitude we turn to a more rigorous method, applying it in the first case to the equation of zero order

$$y'' + \frac{1}{x}y' + y = 0.$$

If we put

$$C_\nu(x) = x^\nu\left\{1 - \frac{x^2}{2(2\nu+2)} + \frac{x^4}{2\,.\,4(2\nu+2)(2\nu+4)} - \ldots\right\},$$

so that incidentally $C_0(x) = J_0(x)$, we know from 4·1(1) that C_ν satisfies Bessel's equation

$$\frac{d^2}{dx^2} C_\nu + \frac{1}{x}\frac{d}{dx} C_\nu + \left(1 - \frac{\nu^2}{x^2}\right) C_\nu = 0.$$

Differentiating this with respect to ν we have

$$\frac{d^2}{dx^2}\left\{\frac{\partial}{\partial\nu} C_\nu\right\} + \frac{1}{x}\frac{d}{dx}\left\{\frac{\partial}{\partial\nu} C_\nu\right\} + \left(1 - \frac{\nu^2}{x^2}\right)\frac{\partial}{\partial\nu} C_\nu = \frac{2\nu}{x^2} C_\nu.$$

On equating ν to zero, the right side vanishes. We conclude that

$$y = \left\{\frac{\partial}{\partial\nu} C_\nu(x)\right\}_{\nu=0}$$

is a solution of Bessel's equation of order zero. It remains to find the explicit form of this. Write $C_\nu = x^\nu S$ where S stands for the series in brackets. We then have

$$\frac{\partial}{\partial\nu} C_\nu = x^\nu S \log x + x^\nu \frac{\partial S}{\partial\nu}$$

$$= C_\nu \log x + x^\nu \frac{\partial S}{\partial\nu}.$$

On equating ν to zero we have

$$\left\{\frac{\partial}{\partial\nu} C_\nu(x)\right\}_{\nu=0} = J_0(x)\log x + \left\{\frac{\partial S}{\partial\nu}\right\}_{\nu=0}.$$

It is left to the curious to verify by logarithmic differentiation that

$$\left\{\frac{\partial S}{\partial\nu}\right\}_{\nu=0} = (\tfrac{1}{2}x)^2 - \frac{(\frac{1}{2}x)^4}{(2!)^2}(1 + \tfrac{1}{2}) + \frac{(\frac{1}{2}x)^6}{(3!)^2}(1 + \tfrac{1}{2} + \tfrac{1}{3}) - \ldots .$$

The result

$$Y_0(x) = J_0(x)\log x + \left\{(\tfrac{1}{2}x)^2 - \frac{(\frac{1}{2}x)^4}{(2!)^2}(1 + \tfrac{1}{2}) + \ldots\right\}$$

is usually associated with the name of Neumann. The full solution of the equation of order zero is

$$y = AJ_0(x) + BY_0(x)$$

and no other solution can have any form but this. We may here insert a remark, of no practical consequence, that in a particular form associated with the name of Weber the values

$$A = -\frac{2}{\pi}(\log 2 - \gamma); \quad B = \frac{2}{\pi}$$

are taken. The number γ is known as Euler's constant; its value is 0·5772 . . . and it is defined as the limit of

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} - \log n$$

as n approaches infinity.

6·3. Equation of integral order.

Having substantiated that the second solution of zero order contains a logarithmic term, we now proceed to render probable the same for integral orders. We can write

$$J_n(x) = \beta x^n(1 + a_1x^2 + a_2x^4 + \ldots),$$

where β, a_1, a_2, &c., are known constants. We then have an expansion of the form

$$\frac{1}{J_n(x)} = \beta^{-1}x^{-n}(1 + b_1x^2 + b_2x^4 + \ldots).$$

Hence

$$\frac{1}{x\{J_n(x)\}^2} = \frac{1}{\beta^2x^{2n+1}}(1 + c_1x^2 + c_2x^4 + \ldots).$$

The $(n + 1)$th term is a multiple of x^{-1} and hence on integration we get a logarithmic term. If we use the series value of $J_n(x)$ in the second solution,

$$y = J_n(x)\int\frac{dx}{x\{J_n(x)\}^2},$$

we conclude that the form is $AJ_n(x)\log x + S_1 + S_2$, where

A = constant,

S_1 = a finite series of negative powers,

S_2 = an infinite series of positive powers.

We do not propose to find the forms of any of these since they would serve us no useful purpose. The result is denoted by $Y_n(x)$ and it becomes infinite when x is zero (fig. 7). The full solution of Bessel's equation of integral order is

$$y = AJ_n(x) + BY_n(x),$$

and every solution must be of this type.

The weakness of the above demonstration lies in our failure to establish the presence of the multiple of x^{-1}. However, as the series S_1 exists, we are still justified in saying that $Y_n(x)$ tends to infinity as x tends to zero. Instead of pursuing the matter it is more profitable to adopt a new outlook.

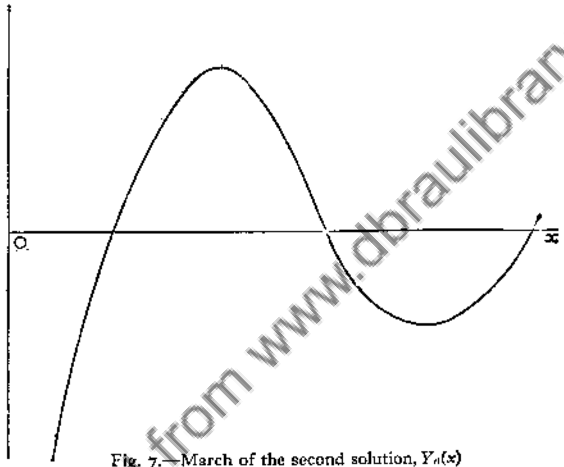

Fig. 7.—March of the second solution, $Y_n(x)$

Let $J_\nu(x)$ be a solution of Bessel's equation of non-integral order ν. By differentiation with respect to ν we have

$$\frac{d^2}{dx^2}\left\{\frac{\partial}{\partial \nu} J_\nu\right\} + \frac{1}{x}\frac{d}{dx}\left\{\frac{\partial}{\partial \nu} J_\nu\right\} + \left(1 - \frac{\nu^2}{x^2}\right)\frac{\partial}{\partial \nu} J_\nu = \frac{2\nu}{x^2} J_\nu.$$

Since $J_{-\nu}(x)$ is also a solution we have similarly

$$\frac{d^2}{dx^2}\left\{\frac{\partial}{\partial \nu} J_{-\nu}\right\} + \frac{1}{x}\frac{d}{dx}\left\{\frac{\partial}{\partial \nu} J_{-\nu}\right\} + \left(1 - \frac{\nu^2}{x^2}\right)\frac{\partial}{\partial \nu} J_{-\nu} = \frac{2\nu}{x^2} J_{-\nu}.$$

Multiply the second of these by $(-1)^n$, where n is an integer, and subtract. In order to avoid piling up formidable masses of symbolism, use the abbreviation

$$F_\nu = \frac{\partial}{\partial \nu} J_\nu - (-1)^n \frac{\partial}{\partial \nu} J_{-\nu}.$$

We then have

$$\frac{d^2F_\nu}{dx^2}+\frac{1}{x}\frac{dF_\nu}{dx}+\left(1-\frac{\nu^2}{x^2}\right)F_\nu=\frac{2\nu}{x^2}\{J_\nu-(-1)^nJ_{-\nu}\}.$$

We notice that if the non-integral ν tends to the integral value n, the right side tends to zero. Hence if F_n denotes what F_ν becomes in similar circumstances, we conclude that F_n is a solution of Bessel's equation, i.e.

$$F_n=Lt\left\{\frac{\partial}{\partial\nu}J_\nu(x)-(-1)^n\frac{\partial}{\partial\nu}J_{-\nu}(x)\right\},\quad \nu\to n.$$

The reason for the foregoing rather abstruse discussion is partly the matter of tabulation and partly dissatisfaction with having the second solution differently defined according as the order is integral or not. The logical thing to do is to adopt a form of the second solution that holds in all circumstances. Various writers have done this, and the corresponding functions usually go by their names. The only one that need concern us is Weber's,

(1) $$Y_\nu(x)=\frac{\cos\nu\pi J_\nu(x)-J_{-\nu}(x)}{\sin\nu\pi}.$$

For non-integral orders this is evidently a solution since it is a linear combination of J_ν and $J_{-\nu}$. For integral orders, $\cos n\pi=(-1)^n$, $\sin n\pi=0$, so that the fraction takes the indeterminate form 0/0. Resolving this in the usual way by differentiating numerator and denominator with respect to ν, we have

$$\frac{-\pi\sin\nu\pi J_\nu+\left\{\cos\nu\pi\frac{\partial}{\partial\nu}J_\nu-\frac{\partial}{\partial\nu}J_{-\nu}\right\}}{\pi\cos\nu\pi}.$$

The result shows that as ν approaches n the fraction becomes F_n/π and is accordingly a solution of Bessel's equation.

Much of the foregoing can be laid aside. All that matters in practice is that there is a second solution $Y_n(x)$ that becomes infinite at the origin. It may be tabulated under the name of Weber or Neumann just as logarithms may be tabulated under the name of Briggs or Napier. That is immaterial so long as they are not confounded in the same calculation. When we meet a Bessel equation of order n, be the order integral or not, we write down the solution as $y=AJ_n+BY_n$, and go on from there.

6·4. Recurrence formulæ.

A consequence of Weber's definition is that Y_n obeys the same recurrence formulæ as J_n, which is a great advantage. If we multiply the formulæ

$$\frac{d}{dx}\{x^\nu J_\nu\} = x^\nu J_{\nu-1}; \quad \frac{d}{dx}\{x^\nu J_{-\nu}\} = -x^\nu J_{-\nu+1}$$

respectively by $\cot \nu\pi$, $\operatorname{cosec} \nu\pi$ and subtract, we have

(1) $$\frac{d}{dx}\{x^\nu Y_\nu\} = x^\nu Y_{\nu-1}.$$

This establishes that Y_ν is a cylinder function and the rest follows.

(2) $$Y_{\nu-1} + Y_{\nu+1} = \frac{2\nu}{x} Y_\nu,$$

(3) $$Y_{\nu-1} - Y_{\nu+1} = 2Y_\nu',$$

(4) $$xY_\nu' + \nu Y_\nu = xY_{\nu-1},$$

(5) $$xY_\nu' - \nu Y_\nu = -xY_{\nu+1},$$

and so on. The proof is based on the assumption of non-integral order; but if we appeal to the principle of continuity we can apply the results to integral orders.

Since Y_ν is a cylinder function, we conclude that it has an infinity of zeros, their interval tending to π; and as $x^{\frac{1}{2}}J_\nu$ and $x^{\frac{1}{2}}Y_\nu$ are two solutions of the same normal form, we see from 2·9 that their zeros must interlace. Moreover, there must be a relation of the form

$$J_\nu Y_\nu' - J_\nu' Y_\nu = \frac{A}{x}$$

(see 3·6, Ex. 10). Using Weber's definition of Y_ν and its differential coefficient, we have

(6) $$J_\nu Y_\nu' - J_\nu' Y_\nu = (J_\nu' J_{-\nu} - J_\nu J_{-\nu}') \operatorname{cosec} \nu\pi$$
$$= \frac{2}{\pi x},$$

the last step being from 4·4(1). It must be pointed out that the right side may take a different value if some second solution other than Weber's is used. This has to be borne in mind when consulting other books.

EXERCISES

1. Establish the following results, which frequently occur in physical problems:

$$(1)\ J_n Y_{n+1} - J_{n+1} Y_n = -\frac{2}{\pi x},$$

$$(2)\ J_n Y_n'' - J_n'' Y_n = -\frac{2}{\pi x^2},$$

$$(3)\ J_n' Y_n'' - J_n'' Y_n' = \frac{2}{\pi x}\left(1 - \frac{n^2}{x^2}\right),$$

$$(4)\ J_n' Y_n''' - J_n''' Y_n' = \frac{2}{\pi x^2}\left(\frac{3n^2}{x^2} - 1\right).$$

2. $$\frac{d}{dx}\{x^{\frac{1}{2}n} Y_n(kx^{\frac{1}{2}})\} = \tfrac{1}{2}kx^{\frac{1}{2}(n-1)} Y_{n-1}(kx^{\frac{1}{2}}),$$

$$\frac{d}{dx}\{x^{-\frac{1}{2}n} Y_n(kx^{\frac{1}{2}})\} = -\tfrac{1}{2}kx^{-\frac{1}{2}(n+1)} Y_{n+1}(kx^{\frac{1}{2}}).$$

3. $$\frac{d}{dx}\{x^{n\alpha} Y_n(kx^{\alpha})\} = k\alpha x^{(n+1)\alpha-1} Y_{n-1}(kx^{\alpha}),$$

$$\frac{d}{dx}\{x^{-n\alpha} Y_n(kx^{\alpha})\} = -k\alpha x^{(1-n)\alpha-1} Y_{n+1}(kx^{\alpha}).$$

4. $$Y_{\frac{1}{2}}(x) = -J_{-\frac{1}{2}}(x) = -\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\cos x,$$

$$Y_{-\frac{1}{2}}(x) = J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\sin x.$$

6·5. The lengthening pendulum (resumed).

Now that we are in possession of a second solution, we can continue with the applications. We revert to the problem of the lengthening pendulum. The equation of motion, 5·1(1),

$$\frac{d^2\theta}{dx^2} + \frac{2}{x}\frac{d\theta}{dx} + \frac{k^2}{x}\theta = 0,$$

has the solution

$$\theta = Ax^{-\frac{1}{2}}J_1(2kx^{\frac{1}{2}}) + Bx^{-\frac{1}{2}}Y_1(2kx^{\frac{1}{2}}),$$

whence

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} = -\frac{Ak}{x}J_2(2kx^{\frac{1}{2}}) - \frac{Bk}{x}Y_2(2kx^{\frac{1}{2}}).$$

With two constants at our disposal we can satisfy two initial conditions. Suppose we take the origin of time when the angular velocity is temporarily zero. Let the corresponding angular displacement be β. We then have

$$t = 0, \quad r = a, \quad \theta = \beta,$$
$$\frac{d\theta}{dt} = 0, \quad x = \frac{a}{b}, \quad k = \left(\frac{g}{b}\right)^{\frac{1}{2}}.$$

For the determination of A, B we have

$$\beta\left(\frac{a}{b}\right)^{\frac{1}{2}} = AJ_1(\lambda) + BY_1(\lambda), \quad \lambda = \frac{2}{b}(ag)^{\frac{1}{2}},$$
$$0 = AJ_2(\lambda) + BY_2(\lambda).$$

In virtue of the relation 6·4, Ex. 1,

$$J_1(\lambda)Y_2(\lambda) - J_2(\lambda)Y_1(\lambda) = -\frac{2}{\pi\lambda},$$

we deduce

$$A = -\tfrac{1}{2}\pi\lambda\beta\left(\frac{a}{b}\right)^{\frac{1}{2}}Y_2(\lambda), \quad B = \tfrac{1}{2}\pi\lambda\beta\left(\frac{a}{b}\right)^{\frac{1}{2}}J_2(\lambda).$$

Hence the angular displacement at any subsequent time is given by

$$\frac{2\theta}{\pi\lambda\beta}\left(\frac{bx}{a}\right)^{\frac{1}{2}} = J_2(\lambda)Y_1(2kx^{\frac{1}{2}}) - Y_2(\lambda)J_1(2kx^{\frac{1}{2}}).$$

EXERCISES

1. Prove that the times of zero displacement are given by the roots of the equation

$$\frac{J_2(\lambda)}{Y_2(\lambda)} = \frac{J_1(2kx^{\frac{1}{2}})}{Y_1(2kx^{\frac{1}{2}})},$$

and that the instants of temporary rest are given by the roots of the equation

$$\frac{J_2(\lambda)}{Y_2(\lambda)} = \frac{J_2(2kx^{\frac{1}{2}})}{Y_2(2kx^{\frac{1}{2}})}.$$

What is the significance of the obvious root, $\lambda = 2kx^{\frac{1}{2}}$?

2. Investigate the problem taking the origin of time at the instant when the bob is passing the vertical with angular velocity ω.

3. The angular displacement and velocity are given in the text. Calculate the angular acceleration, substitute all three in the equation of motion and deduce the recurrence formula.

4. The radial velocity is known to be constant; calculate the transverse velocity.

5. In what circumstances would the solution depend on the function Y without J?

6. Prove that the angular displacement and the lineal lateral displacement do not reach maxima simultaneously.

7. Assign two reasons why the successive maximum angular displacements decrease. [The case of lineal displacements has to be left open for the moment; it depends on the ultimate behaviour of the cylinder function $x^{\frac{1}{2}}C_1(x)^{\frac{1}{2}})$.]

8. Prove that the constant A has the physical dimensions $T^{\frac{1}{2}}$.

6·6. Motion of a variable mass.

Problem 5.—The rectilineal motion of a variable mass under a variable force.

Suppose a mass moves along OX under an attraction to the origin. If the force per unit mass is proportional to the distance and we equate it to the rate of change of linear momentum, we have

$$\frac{d}{dt}\left(m\frac{dx}{dt}\right) = -c^2mx,$$

where c^2 is the constant of proportionality. Hence

$$\frac{d^2x}{dt^2} + \frac{1}{m}\frac{dm}{dt}\cdot\frac{dx}{dt} + c^2x = 0.$$

Suppose further that the mass suffers abrasion or any form of attenuation so that its magnitude at time t is $m = (a + bt)^{-1}$. Thus the mass, initially a^{-1}, is asymptotic to zero. Then by logarithmic differentiation,

$$\frac{1}{m}\frac{dm}{dt} = -\frac{b}{a+bt},$$

and the equation of motion becomes

$$\frac{d^2x}{dt^2} - \frac{b}{a+bt}\frac{dx}{dt} + c^2x = 0.$$

Make the substitution

$$a + bt = bz, \quad dt = dz,$$

and we have

$$\frac{d^2x}{dz^2} - \frac{1}{z}\frac{dx}{dz} + c^2x = 0.$$

Comparison with our general form 3·6(2) gives

$$1-2\alpha=-1,\quad n\gamma=\alpha,\quad 2(\gamma-1)=0,\quad \beta\gamma=c,$$

whence

$$\alpha=1=\gamma=n,\quad \beta=c.$$

The solution is

$$x=AzJ_1(cz)+BzY_1(cz),$$

and the motion is oscillatory. The velocity at any instant is given by

$$\frac{dx}{dt}=\frac{dx}{dz}=AczJ_0(cz)+BczY_0(cz).$$

To determine the two constants from assigned conditions, let us suppose that when t is zero the mass is at temporary rest at distance h from the origin. This gives $z_0=a/b$ and

$$AJ_0(\lambda)+BY_0(\lambda)=0,\qquad \lambda=\frac{ca}{b},$$

$$AJ_1(\lambda)+BY_1(\lambda)=\frac{hb}{a}.$$

In virtue of the relation 6·4, Ex. 1,

$$J_1(\lambda)Y_0(\lambda)-J_0(\lambda)Y_1(\lambda)=\frac{2}{\pi\lambda}=\frac{2b}{\pi ca},$$

these solve to

$$A=\tfrac{1}{2}\pi chY_0(\lambda),\qquad B=-\tfrac{1}{2}\pi chJ_0(\lambda).$$

Suppose we require the time to reach the origin; then $x=0$ gives

$$\frac{J_1(cz)}{Y_1(cz)}=-\frac{B}{A}=\frac{J_0(\lambda)}{Y_0(\lambda)}.$$

The fraction on the right is then a definite number, s say. Hence

$$J_1(cz)-sY_1(cz)=0.$$

Here the left side is a cylinder function and there is therefore an infinite number of suitable values of cz. Their interval is greater than π and gradually tends to it. The time interval between successive transits through the origin tends to the value π/c, as in simple harmonic motion. In view of the fact that the force per unit mass is proportional to the distance, it would be interesting to see an argument, based on first principles, explaining the apparent anomaly that the motion is not simple harmonic. The total mass varies; but the total force varies in the same ratio. Having disposed of that trifle, follow it up by explaining why the amplitude continually increases.

To prove this last statement, multiply the equation of motion

$$\ddot{x} - \frac{b\dot{x}}{a + bt} + c^2 x = 0$$

by $2\dot{x}$ and integrate. This gives

$$\Big[\dot{x}^2\Big]_\alpha^\beta - 2b\int_\alpha^\beta \frac{\dot{x}^2}{a + bt}\,dt + \Big[c^2x^2\Big]_\alpha^\beta = 0.$$

Taking α, β as times of temporary rest, not necessarily consecutive, we dispose of the first term. The integrand is certainly positive and hence x is greater at β than at α. The oscillation increases in amplitude, like a vibration picking up on resonance.

Another curious feature of the motion is that the times for half-swings inward to the origin continually increase; but the times for half-swings away from the origin continually decrease. This is the problem referred to in 4·6, Ex. 23. To prove it, let 0, c_1, c_2, &c., denote the values of cz for zero velocity, and let d_1, d_2, &c., denote the values for zero displacement. We then have

$$d_{r+1} - d_r > \pi, \quad c_r - c_{r-1} < \pi,$$

whence by subtraction

$$(d_{r+1} - c_r) - (d_r - c_{r-1}) > 0, \quad d_{r+1} - c_r > d_r - c_{r-1}.$$

Similarly from

$$c_r - c_{r-1} < \pi, \quad d_r - d_{r-1} > \pi,$$

we deduce

$$(c_r - d_r) - (c_{r-1} - d_{r-1}) < 0, \quad c_r - d_r < c_{r-1} - d_{r-1}.$$

A rough sketch shows that this establishes our contention.

EXERCISES

1. Derive the acceleration by differentiating the velocity and check by the equation of motion.

2. In simple harmonic motion, when the displacement is zero the velocity is maximum and the acceleration changes sign. Prove that in the present problem this does not hold.

Also in simple harmonic motion the displacement and the acceleration reach their maxima simultaneously. Does this hold in the present case?

3. Determine the constants if initially the body is passing through the origin with velocity u.

4. According to the principles of dynamics, the momentum of the body when passing through the origin equals the time integral of the force since the previous rest position. How does this work out in terms of the Bessel functions?

5. Examine the problem when the mass at time t is $(a + bt)^{-3}$.

6. Prove that the kinetic energy of the body when passing the origin is proportional to $\alpha\{AJ_0(\alpha) + BY_0(\alpha)\}^2$, where α is a value of cz corresponding to $x = 0$.

6·7. Longitudinal vibrations of a bar.

Problem 6.—Discuss the longitudinal vibrations in a tapered bar.

Imagine a long thin metal bar to occupy a finite part of the x-axis from the origin. As the result of a longitudinal blow, the section A which was originally at distance x is displaced to distance $x + u$. Note that u is not necessarily small since the bar may have moved bodily. Similarly the section B, originally at $x + \delta x$, is displaced a distance $u + \delta u$. The extension of the element δx is thus δu, so that the strain is $\partial u/\partial x$ and the stress is $E\,\partial u/\partial x$, where E is Young's modulus. The convention of sign is fixed by the stress being tensile when δu is positive.

If a be the area of cross-section A, and F the force acting on it, we have

$$F = aE\frac{\partial u}{\partial x}.$$

Similarly the force acting on the section B is $F + \delta F$. The resultant to the right is

$$\delta F = \frac{\partial F}{\partial x}\,\delta x = \frac{\partial}{\partial x}\left\{aE\frac{\partial u}{\partial x}\right\}\delta x.$$

The mass of the element AB is $\rho a\,\delta x$ if ρ is the density; and as its acceleration is $\partial^2 u/\partial t^2$ we have the equation of motion

$$\rho a\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left\{aE\frac{\partial u}{\partial x}\right\}.$$

The foregoing assumes that all points of any normal cross-section simultaneously suffer the same displacement. It is usually customary further to assume that E, ρ and a are constants. The equation then reduces to

$$\frac{\partial^2 u}{\partial t^2} = k^2\frac{\partial^2 u}{\partial x^2}; \qquad k^2 = \frac{E}{\rho}.$$

This equation has solutions of the type

$$u = A \frac{\cos}{\sin} nx \frac{\cos}{\sin} knt.$$

The choice of sine or cosine is determined by the mode of support. The length of the bar then determines n; and the initial conditions determine A. Thus if both ends of the bar of length λ are fixed, so that $u = 0$ both when $x = 0$ and when $x = \lambda$, a suitable solution is

$$u = \Sigma \sin nx(A \cos knt + B \sin knt),$$

provided that $\sin n\lambda = 0$. This gives $n\lambda = \pi, 2\pi, 3\pi, \ldots$. The permissible values of n are thus determined, and the constants A, B are chosen to suit given initial conditions of motion and displacement.

The above cursory discussion does not involve Bessel functions and fuller treatment on these lines is to be found in texts on sound. If we abandon the uniformity of section whilst still treating E and ρ as constants, the equation of motion becomes

$$\frac{\rho}{E}\frac{\partial^2 u}{\partial t^2} = \frac{1}{a}\frac{\partial}{\partial x}\left\{a\frac{\partial u}{\partial x}\right\}.$$

On the grounds that longitudinal vibration is possible we write $u = X \sin(pt + \phi)$, where X is independent of t. The equation becomes

$$\frac{d^2X}{dx^2} + \frac{1}{a}\frac{da}{dx}\frac{dX}{dx} + k^2X = 0, \quad k^2 = \frac{p^2\rho}{E}.$$

If the bar is slightly tapered we can write

$$a = \alpha\left(1 + \frac{x}{\beta}\right), \quad \frac{da}{dx} = \frac{\alpha}{\beta}, \quad \frac{1}{a}\frac{da}{dx} = \frac{1}{x + \beta},$$

whence

$$\frac{d^2X}{dx^2} + \frac{1}{x+\beta}\frac{dX}{dx} + k^2X = 0.$$

With the slight modification

$$x + \beta = z, \quad dx = dz,$$

we have

$$\frac{d^2X}{dz^2} + \frac{1}{z}\frac{dX}{dz} + k^2X = 0,$$

so that

$$X = AJ_0(kz) + BY_0(kz).$$

As an illustration we suppose the left end to be fixed and the right end to be free. At the left we have the conditions

$$u = 0 = X = x, \quad z = \beta.$$

Hence

$$0 = AJ_0(k\beta) + BY_0(k\beta).$$

At the free end there can be no stress and we have the conditions

$$x = \lambda, \quad z = \lambda + \beta, \quad \frac{\partial u}{\partial x} = 0 = \frac{dX}{dx} = \frac{dX}{dz},$$

whence

$$0 = AJ_1\{k(\lambda + \beta)\} + BY_1\{k(\lambda + \beta)\}.$$

The free end of the bar, a place of zero stress, is a place of maximum movement, a loop. The foregoing is in line with the fact that the maxima of J_0 and Y_0 are located by the zeros of J_1 and Y_1 respectively. The elimination of the ratio A/B gives

$$\frac{J_0(k\beta)}{Y_0(k\beta)} = \frac{J_1\{k(\lambda + \beta)\}}{Y_1\{k(\lambda + \beta)\}}.$$

The first root of this equation in k corresponds to a stress distribution which is part of a loop of the curve $AJ_1(kz) + BY_1(kz)$, with a zero at the right. The higher roots give distributions with one or more stress-nodes, there being always one at the right. In each case the stress at any point varies sinusoidally with the time.

EXERCISES

1. If the tapered bar in the text be fixed at both ends, prove that the stress distribution is determined by the roots of the equation

$$\frac{J_0(k\beta)}{Y_0(k\beta)} = \frac{J_0\{k(\lambda + \beta)\}}{Y_0\{k(\lambda + \beta)\}}.$$

2. If the law of taper is $a = \alpha(1 + x/\beta)^{-1}$, prove that the solution of the fixed-free bar depends on

$$\frac{J_1(k\beta)}{Y_1(k\beta)} = \frac{J_0\{k(\lambda + \beta)\}}{Y_0\{k(\lambda + \beta)\}}.$$

3. Prove that the problem of the longitudinal vibrations in a tapered bar is soluble by Bessel functions when the law of tapering is $a = \alpha(1 + x/\beta)^m$, where m is any real number, positive or negative.

Investigate the case where $m = \frac{1}{2}$.

6·8. The vibrating membrane.

Problem 7.—The vibrations of a stretched membrane.

A concise history of English literature could be pardoned for omitting the name of John Clare; but the complete omission of Shakespeare could hardly be condoned. Similarly from a survey of the relevant literature it appears that even the most cursory treatment of Bessel functions cannot avoid the vibrating membrane. It was first discussed nearly two centuries ago by L. Euler in 1764; it has been heavily belaboured in innumerable textbooks ever since.

The reader is aware that a stretched membrane, such as a drum or tambourine, is capable of vibrating when tapped. The problem is the two-dimensional analogue of the vibrating string and similar assumptions are made for effecting a solution. A membrane differs from a disc as a string differs from a rod in that flexural rigidity, shear and bending are taken as negligible. The vibrations are presumed due to the tension applied to the material.

The equilibrium position of the membrane is taken as the horizontal plane and gravity is ignored. During movement let z be the upward displacement of a point whose co-ordinates are otherwise r, θ. Consider the element limited by the arcs of radii r and $r + \delta r$, and the radii defined by θ and $\theta + \delta\theta$. If T be the tension per unit length along the edge $r\,\delta\theta$, the force on this edge is $F = Tr\,\delta\theta$ acting at an angle ψ to the horizontal. The vertically downward component of this is

$$Y = F\sin\psi, \quad \sin\psi = \frac{\partial z}{\partial r}.$$

At the opposite edge the vertically upward component is $Y + \delta Y$ and the upward resultant is

$$\delta Y = \frac{\partial Y}{\partial r}\,\delta r = \frac{\partial}{\partial r}\left(Tr\,\frac{\partial z}{\partial r}\right)\delta r\,\delta\theta.$$

Along the edge δr we similarly have

$$F_1 = T_1\delta r, \quad Y_1 = F_1\sin\psi_1, \quad \sin\psi_1 = \frac{1}{r}\frac{\partial z}{\partial\theta}.$$

The upward resultant from the two δr edges is

$$\frac{\partial Y_1}{\partial\theta}\,\delta\theta = \frac{\partial}{\partial\theta}\left\{\frac{T_1}{r}\frac{\partial z}{\partial\theta}\right\}\delta\theta\,\delta r.$$

If ρ be the surface density, the mass of the element is $\rho r\,dr\,d\theta$ and its upward acceleration is $\partial^2 z/\partial t^2$. As in the case of a string, and for the same reason, it is customary to take the tension as constant; this gives the equation of motion

$$\frac{\partial}{\partial r}\left\{r\frac{\partial z}{\partial r}\right\}+\frac{1}{r}\frac{\partial^2 z}{\partial\theta^2}=\frac{\rho r}{T}\frac{\partial^2 z}{\partial t^2}.$$

As z is presumed to undergo periodic variations, expressible in a Fourier series, we write

$$z=w\exp(ipt),\quad \frac{\partial^2 z}{\partial t^2}=-p^2 w\exp(ipt),$$

where w is independent of t. This gives

$$\frac{\partial}{\partial r}\left\{r\frac{\partial w}{\partial r}\right\}+\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2}+k^2 rw=0,\quad k^2=\frac{p^2\rho}{T}.$$

We now further assume $w=u\cos n\theta$, where u is independent of θ and n is an integer. The significance of this assumption will appear later. We now have

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{du}{dr}\right\}+\left(k^2-\frac{n^2}{r^2}\right)u=0,$$

of which the solution is

$$u=AJ_n(kr)+BY_n(kr).$$

Accordingly, for the displacement,

$$z=\frac{\cos}{\sin}pt\cos n\theta\{AJ_n(kr)+BY_n(kr)\}.$$

So far nothing has been said about the shape of the membrane; but the analysis is obviously adapted to the circular form, and this will be employed. If the membrane is complete up to the centre, as in a tambourine, the function Y_n must be discarded since it makes z infinite at the origin. It may be retained for an annular membrane.

In the simplest case, where n is zero, we can take $z=A\cos pt J_0(kr)$. As z is permanently zero at the edge, where $r=a$, we have $J_0(ka)=0$. The smallest permissible value of ka is 2·4048. For a given membrane, this determines k and thence p and the corresponding frequency.

It is known that $J_0(x)$ has an infinity of zeros. We denote the succession by $c_1, c_2, \ldots$, and the corresponding values of k by $k_1, k_2, \ldots$, so that $k_s a=c_s$. A curious consequence of this can be illustrated by giving s the value 3, say. If z is proportional to $J_0(k_3r)$, then at the boundary where $r = a$ we have z proportional to

$$J_0(k_3a) = J_0(c_3) = 0.$$

But nearer the centre, where $r = k_1a/k_3$, we have z proportional to

$$J_0(k_3r) = J_0(k_1a) = J_0(c_1) = 0.$$

The conclusion is that this value of r gives a nodal circle; and the same applies if $r = k_2a/k_3$. In general there are $(s - 1)$ nodal circles if $J_0(k_sa) = 0$. This is the analogue of the possible nodes on a vibrating string.

In the more general case where z is proportional to $\cos n\theta J_n(kr)$ the displacement is permanently zero if either $J_n(kr) = 0$ or $\cos n\theta = 0$. The former must hold at the periphery and $J_n(ka) = 0$. This has an infinite number of roots, each of which gives a corresponding value of k. There are no nodal circles, except the boundary, for the smallest root; the higher roots give nodal circles. The alternative $\cos n\theta = 0$ gives nodal diameters corresponding to $n\theta = \frac{1}{2}\pi, \frac{3}{2}\pi, \ldots$.

In the above solution, all points simultaneously pass through the equilibrium position and hence have the same period. This is known as a "normal mode"; in general the motion is more complicated than this. In a normal mode, with nodal circles and diameters, adjacent sectors are in opposite phase. The matter is treated more fully in texts on sound.

EXERCISES

1. If an annular membrane has inner and outer radii a, b respectively, prove that its slowest mode of vibration is given by the smallest root of the equation

$$\frac{J_0(ka)}{Y_0(ka)} = \frac{J_0(kb)}{Y_0(kb)}.$$

2. Reverting to the problem of the taut string, a non-uniform string of length λ has its ends fixed and is kept taut by a tension T. The line-density at distance x from one end is $\rho(1 + kx)^{-1}$. With the notation of the text, prove that the possible frequencies are given by

$$\frac{J_1(c)}{Y_1(c)} = \frac{J_1(nc)}{Y_1(nc)}, \quad n = (1 + k\lambda)^{\frac{1}{2}}, \quad c = \frac{2p}{k}\left(\frac{\rho}{T}\right)^{\frac{1}{2}}.$$

3. Investigate the more general case of the vibrating string when the law of density is $\rho(1 + kx)^{-\alpha}$, $\alpha = 2 - m^{-1}$, m an integer.

6·9. The oscillating chain.

Problem 8.—*The small oscillations of a uniform chain suspended at one end.*

This is another of the classical problems of our subject. It was first discussed in 1732 by Daniel of the Swiss family Bernoulli; later on in 1781 it was taken up by Euler. The chain is a flexible filament of uniform line-density, devoid of flexural rigidity. Let one end be fixed to the origin O, with OY to the right and OX downwards. If A be a point defined by $OA = s$, the tension T at A makes angle ψ with OX and has an inward component given by $Y = T \sin\psi$. At an adjacent point B defined by $s + \delta s$ the corresponding outward component is $Y + \delta Y$, so that the outward resultant on the element AB is

$$\delta Y = \frac{\partial Y}{\partial s}\,\delta s = \frac{\partial}{\partial s}\left\{T\,\frac{\partial y}{\partial s}\right\}\delta s.$$

If ρ be the line density, the mass of the element AB is $\rho\,\delta s$ and its acceleration is $\ddot{y}$; moreover, since the oscillations are small, it is sufficiently accurate to take the tension T as the weight of chain below A. Hence $T = \rho g(\lambda - s)$, where λ is the length of the chain. This gives as the equation of motion

$$\rho\,\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial s}\left\{\rho g(\lambda - s)\,\frac{\partial y}{\partial s}\right\}.$$

The oscillatory nature of the motion can be allowed for by the substitution $y = u\cos(pt + \phi)$ and the equation for u becomes

$$(\lambda - s)\,\frac{d^2u}{ds^2} - \frac{du}{ds} + \frac{p^2}{g}\,u = 0.$$

The substitution $\lambda - s = z$, $ds = -dz$, which is equivalent to measuring from the lower end of the chain, gives

$$\frac{d^2u}{dz^2} + \frac{1}{z}\,\frac{du}{dz} + \frac{k^2u}{z} = 0, \quad k^2 = \frac{p^2}{g}.$$

The solution is obtained by comparison with our general standard 3·6(2) and yields

$$a = 0, \quad \gamma = \tfrac{1}{2}, \quad n = 0, \quad \beta = 2k.$$

Thus

$$u = AJ_0(2kz^{\frac{1}{2}}) + BY_0(2kz^{\frac{1}{2}}).$$

For the determination of the constants we examine the free end. Here $s = \lambda$, $z = 0$ and as the displacement y (and therefore u) is to be small we must have B zero. The other constant remains indeterminate since the oscillations are merely small. At the fixed end O we have $y = 0 = u = s$, $z = \lambda$ so that $J_0(2k\lambda^{\frac{1}{2}}) = 0$. Thus $2k\lambda^{\frac{1}{2}}$ is a zero of $J_0(x)$; any particular zero furnishes a value of k and this in turn determines p and the corresponding frequency of oscillation.

If we equate t to zero to get the initial shape of the chain we conclude that, for the simple type of oscillation we have been discussing, the chain must be started in the form of an odd number of half-loops of the curve $J_0(2kz^{\frac{1}{2}})$ with a zero at O. By slightly changing the origin of time the system can start from rest; but the form will still have to be as stated.

EXERCISES

1. Examine the case of the non-uniform chain where the density is proportional to the mth power of the distance from the free end. Prove that the solution depends on the function $z^{-\frac{1}{2}m}J_m(\beta z^{\frac{1}{2}})$, $\beta^2 g = 4p^2(m + 1)$. It was pointed out by Greenhill that a practical approximation would be a large number of parallel wires of non-uniform length connected on the principle of the Venetian blind.

2. If the end of the chain carries a mass, equivalent to a length h, prove that the normal modes are given by

$$\frac{J_2(2kh^{\frac{1}{2}})}{Y_2(2kh^{\frac{1}{2}})} = \frac{J_0(2kl^{\frac{1}{2}})}{Y_0(2kl^{\frac{1}{2}})},$$

where $l = \lambda + h$.

6·10. Heat conduction in one dimension.

One of the outstanding characteristics of physical constants is that almost inevitably they tend to show variation over an extended range. Thus a coefficient of expansion which normally has the constant value a is pretty certain to need modification if it is to be employed over a wide range of temperatures. The form to be adopted is then empirical, and even if theory indicates the nature of the departure from normal, experimental verification would be needed before adopting any specific form. The form most often adopted is a simple polynomial $a + \beta t + \gamma t^2$, the number of coefficients being a measure of one's fastidiousness, and their values being determined by some such method as the principle of least squares. For the most part the above method works quite well and it is usually possible to derive a series solution for any differential equation in which the modified coefficient is employed. The method has one drawback from our point of view, in that Bessel functions fail to be of any assistance.

Since the law is empirical there is little lost and a substantial gain in dropping the above quadratic form in favour of $a(1 + kt)^n$. Both have three constants, which is usually enough for even the captious, and they can be determined to give minimum error over a desired range; but the latter form has the advantage that it fits in with the form of Bessel's equation. This accounts for its presence in such widely different researches as the tapered transmission line on the one hand, and the transverse vibrations of tapered rods on the other. Gauss has given a method of determining the best places to measure when a variable quantity is to be determined from three (or any limited number) measurements, and the three-constant formula is of wide application.

Problem 9.—One-dimensional heat conduction in a heterogeneous medium.

Consider a slab of material bounded by two parallel plane faces (a wall) and suppose that the temperature is uniform over any plane parallel to the faces but varies from one plane to another. The flow of heat is then normal to the planes. Consider an area a parallel to the planes. The heat flow H per second through this area varies jointly as the area, the conductivity κ and the negative temperature gradient. If θ denote the temperature at any point, this gives

$$H = -\kappa a \frac{\partial \theta}{\partial x},$$

where the x-axis is normal to the faces and the origin can be taken at any convenient point. At an adjacent place defined by $x + \delta x$ the heat flow is $H + \delta H$, so that the heat accession to the volume element is

$$-\delta H = -\frac{\partial H}{\partial x}\,\delta x = +\frac{\partial}{\partial x}\left\{\kappa a \frac{\partial \theta}{\partial x}\right\}\delta x.$$

This will show itself in a temperature change. The volume of the element is $a\,\delta x$, so that its heat capacity is $a\rho s\,\delta x$, where ρ is the density and s the specific heat. We accordingly have

$$\frac{\partial}{\partial x}\left\{\kappa a \frac{\partial \theta}{\partial x}\right\}\delta x = a\rho s \frac{\partial \theta}{\partial t}\,\delta x.$$

The area a being the same at both ends of the element, we deduce

$$\frac{\partial^2\theta}{\partial x^2} + \frac{1}{\kappa}\frac{\partial\kappa}{\partial x}\frac{\partial\theta}{\partial x} = \frac{\rho s}{\kappa}\frac{\partial\theta}{\partial t}$$

as the equation of heat conduction.

We attempt to effect a solution by the substitution $\theta = X\exp(-pt)$. Here X is independent of t; the negative exponential is preferred since a temperature may reasonably be expected to decrease to zero, but not to augment indefinitely. The equation becomes

$$\frac{d^2X}{dx^2} + \frac{1}{\kappa}\frac{d\kappa}{dx}\frac{dX}{dx} + \frac{\rho sp}{\kappa}X = 0.$$

In the simplest case the physical coefficients are all taken to be constants. The middle term then drops out and the equation is soluble in trigonometrical functions. As a first modification let us assume ρ replaced by $f\rho(x+f)^{-1}$. This variable density might be ascribed to any of a number of causes such as porosity, humidity, weathering, and so on. The coefficients being otherwise constant we now have

$$\frac{d^2X}{dx^2} + \frac{b^2}{f+x}X = 0, \quad b^2 = \frac{\rho spf}{\kappa}.$$

The substitution

$$f+x = z, \quad dx = dz$$

gives

$$\frac{d^2X}{dz^2} + \frac{b^2}{z}X = 0.$$

This is soluble by Bessel functions. Comparison with our standard 3·6(2) gives

$$\alpha = \tfrac{1}{2}, \quad \beta = 2b, \quad \gamma = \tfrac{1}{2}, \quad n = 1.$$

The solution is

$$X = Az^{\frac{1}{2}}J_1(2bz^{\frac{1}{2}}) + Bz^{\frac{1}{2}}Y_1(2bz^{\frac{1}{2}}).$$

The temperature is therefore

$$\theta = z^{\frac{1}{2}}e^{-pt}\{AJ_1(2bz^{\frac{1}{2}}) + BY_1(2bz^{\frac{1}{2}})\}.$$

The constants A, B would have to be determined to fit pre-assigned initial conditions, and no doubt a series of values of p would have to be employed. This aspect of the matter will be resumed later.

EXERCISES

1. Calculate the temperature gradient. If one face, defined by $z = \lambda$, is permanently at the temperature of the external medium, deduce that

$$AJ_0(2b\lambda^{\frac{1}{2}}) + BY_0(2b\lambda^{\frac{1}{2}}) = 0.$$

2. If the variable density is $\rho f^2(x+f)^{-2}$ and the variable specific heat is $sh^{-1}(x+h)$, deduce that, with the notation of the text,

$$X = z^{\frac{1}{2}}\{AJ_n(2bz^{\frac{1}{2}}) + BY_n(2bz^{\frac{1}{2}})\},$$

where

$$b^2\kappa h = \rho spf^2, \quad 1 + 4b^2c = n^2, \quad c = f - h,$$

so that the order n is real provided $1 + 4b^2c > 0$.

3. Presuming the density and the specific heat to be constants, let the variable conductivity be $\kappa(1 + x/\alpha)^r$.

(i) Prove that the equation is soluble in terms of J_2, Y_2 if $r = 3$.

(ii) If $r = 1$ the functions are of zero order.

(iii) If $r = 1\frac{1}{2}$ the functions are of the first order.

(iv) Prove that the equation is soluble by Bessel functions for all values of r except $r = 2$, in which case the solution can be effected by elementary means.

(v) If $r = 2$ the equation may still be soluble by Bessel functions if ρ and s vary separately or jointly. Investigate this.

4. Investigate the flow of heat along a lagged bar of uniform section on the assumption that the conductivity, density and specific heat are proportional to x^a, x^b and x^c respectively.

6·11. The tapered strut.

Problem 10.—*Discuss the stability of the tapered strut.*

An elementary form of this problem has already been considered in the previous chapter. We continue with the case of the triangular ribbon, first removing the tip and thus reducing it to the trapezoidal form. This gives $I = H(x+h)/h$. The number h defines the fictitious vertex of the triangle. The equilibrium equation is

$$EH\,\frac{(x+h)}{h}\,\frac{d^2y}{dx^2} + Py = 0,$$

or,

$$\frac{d^2y}{dz^2} + \frac{k^2}{4z}\,y = 0, \quad z = x + h, \quad k^2 = \frac{4Ph}{EH}.$$

This has previously been solved and we have

$$y = z^{\frac{1}{2}}\{AJ_1(kz^{\frac{1}{2}}) + BY_1(kz^{\frac{1}{2}})\}.$$

The end conditions being

$$x = 0, \quad z = h,$$

$$x = \lambda, \quad z = h + \lambda = l,$$

we derive

$$0 = AJ_1(kh^{\frac{1}{2}}) + BY_1(kh^{\frac{1}{2}}),$$

$$0 = AJ_1(kl^{\frac{1}{2}}) + BY_1(kl^{\frac{1}{2}}).$$

The ratio A/B is determinate, but one of them remains arbitrary. The elimination of the ratio gives

$$J_1(kh^{\frac{1}{2}})Y_1(kl^{\frac{1}{2}}) = J_1(kl^{\frac{1}{2}})Y_1(kh^{\frac{1}{2}})$$

as the transcendental equation which determines k; this in turn gives P.

As a somewhat less simple illustration we take the left end to be clamped horizontal. The origin is at the left and the y axis is positive downwards. The right end is under horizontal thrust P, but is pinned to prevent lateral displacement. This calls up a lateral force which we may call R and presume to act downwards. An equal and opposite thrust P acts at the clamp, as also an upthrust R. There is further a couple C which is counterclockwise on the assumption that the deflection is downward. Consideration of the moments about any point of the ribbon with co-ordinates x, y gives the equilibrium equation

$$EIy'' = C - Rx - Py.$$

With the above assumption as to the value of I we have

$$EH\frac{(x + h)}{h}\frac{d^2y}{dx^2} = C - Rx - Py.$$

It is statically evident that $C = R\lambda$, and hence

$$\frac{EH}{h}z\frac{d^2y}{dz^2} = R(\lambda + h - z) - Py, \quad z = x + h.$$

There is now a particular integral

$$y = R(\lambda + h - z)/P.$$

The reduced, or auxiliary, equation can be written

$$\frac{d^2y}{dz^2} + \frac{k^2}{4z}y = 0, \quad k^2 = \frac{4Ph}{EH}.$$

The solution is

$$y = z^{\frac{1}{2}}\{AJ_1(kz^{\frac{1}{2}}) + BY_1(kz^{\frac{1}{2}})\} + \frac{R}{P}(\lambda + h - z),$$

whence

$$\frac{dy}{dx} = \frac{dy}{dz} = \tfrac{1}{2}k\{AJ_0(kz^{\frac{1}{2}}) + BY_0(kz^{\frac{1}{2}})\} - \frac{R}{P}.$$

For the end conditions at the left we have

$$x = 0, \quad z = h, \quad y = 0 = y'.$$

Hence

$$0 = AJ_1(kh^{\frac{1}{2}}) + BY_1(kh^{\frac{1}{2}}) + \frac{R\lambda}{Ph^{\frac{1}{2}}},$$

$$0 = AJ_0(kh^{\frac{1}{2}}) + BY_0(kh^{\frac{1}{2}}) - \frac{2R}{Pk}.$$

At the right we have

$$x = \lambda, \quad z = \lambda + h = l, \quad y = 0.$$

Hence

$$0 = AJ_1(kl^{\frac{1}{2}}) + BY_1(kl^{\frac{1}{2}}).$$

These three homogeneous equations can determine nothing more than the ratios $A : B : R$ even if consistent. It is readily verified that

$$\frac{-A}{2h^{\frac{1}{2}}Y_1(kh^{\frac{1}{2}}) + k\lambda Y_0(kh^{\frac{1}{2}})} = \frac{B}{2h^{\frac{1}{2}}J_1(kh^{\frac{1}{2}}) + k\lambda J_0(kh^{\frac{1}{2}})} = \frac{\pi R}{2P}.$$

The condition that the three homogeneous equations are consistent is obtained by eliminating the ratio $A : B$. The result is evidently

$$\frac{2h^{\frac{1}{2}}Y_1(kh^{\frac{1}{2}}) + k\lambda Y_0(kh^{\frac{1}{2}})}{2h^{\frac{1}{2}}J_1(kh^{\frac{1}{2}}) + k\lambda J_0(kh^{\frac{1}{2}})} = \frac{Y_1(kl^{\frac{1}{2}})}{J_1(kl^{\frac{1}{2}})}.$$

This is the transcendental equation that determines k, which in turn determines the critical thrust P.

It is possible to locate the point that tends to maximum deflection by equating y' to zero; but the manipulative algebra involved is more tedious than illuminating. The lateral force R remains indeterminate and the curve taken by the ribbon is reminiscent of the profile of the underside of a spoon. Consider the function $R(\lambda + h - z) - Py$. It is zero at the right; at the left it is positive but decreasing. It accordingly has a zero somewhere on the ribbon and it is not difficult to see that this locates the inflection. The fact that it vanishes at the right is equivalent to the absence of any end-couple.

Reverting to the case of the ribbon under simple end-thrust, let us take

$$I = H\left(\frac{h}{x+h}\right)^r = H\left(\frac{h}{z}\right)^r, \quad z = x + h.$$

The equilibrium equation takes the form

$$y'' + m^2 z^r y = 0, \quad h^r m^2 = \frac{P}{EH}.$$

On comparison with our standard 3·6(2) we deduce

$$\alpha = \tfrac{1}{2}, \quad \gamma = 1 + \tfrac{1}{2}r, \quad n = \frac{1}{2+r}, \quad \beta = \frac{m}{\gamma}.$$

It appears from this that the problem is soluble in terms of Bessel functions for all values of r except -2. As r falls below -1, the order n augments without limit. When r is zero we have $\alpha = \frac{1}{2} = n$, so that the solutions are of the type $x^{\frac{1}{2}}J_{\frac{1}{2}}$. But when r is zero, the ribbon is uniform and the solution is known to be trigonometrical. We thus revert to the theorem that functions of order half an odd integer are expressible trigonometrically.

A more interesting result is derived from letting h become indefinitely large, so that z also becomes large. Since β^2 is proportional to h^{-r}, we have βz^γ ultimately proportional to h, so that the function $J_n(\beta z^\gamma)$ approximates to $J_n(h)$. But as h augments indefinitely, the ribbon approaches uniformity, irrespective of the value of n. The solution is then trigonometrical. We conclude that, for large values of the argument and irrespective of the order, the function $J_n(x)$ between consecutive zeros behaves like $x^{-\frac{1}{2}} \sin x$. The result is one aspect of what are known as the asymptotic formulæ; it marks an advance on our previous knowledge of large values, which was (a) that the absolute magnitude of the stationary value continuously decreases, and (b) that $xJ_1(x)$ increases indefinitely with x.

EXERCISES

1. Taking $I = H(x + h)/h$ for a ribbon of length λ with both ends built-in and subjected to end-thrust P, prove that the condition of stability is

$$\frac{J_0(kh^{\frac{1}{2}})}{Y_0(kh^{\frac{1}{2}})} = \frac{J_0(kl^{\frac{1}{2}})}{Y_0(kl^{\frac{1}{2}})}; \quad k^2 = \frac{4Ph}{EH}; \quad l = \lambda + h.$$

2. The ribbon of length λ is built-in at the left. At the right is vertical load W and horizontal thrust P. Taking

$$I = H\left(\frac{x + h}{h}\right)^{5/3} = H\left(\frac{z}{h}\right)^{5/3},$$

prove that the conditions at the left lead to

$$0 = AJ_3(p) + BY_3(p) + \frac{C}{Ph^{\frac{1}{3}}},$$

$$0 = AJ_2(p) + BY_2(p) - \frac{6Wh^{\frac{1}{3}}}{mP}$$

where

$$m^2 = \frac{36Ph^{5/3}}{EH}, \quad m^6h = p^6.$$

The end condition at the right leads to

$$0 = AJ_3(q) + BY_3(q), \quad m^6l = q^6, \quad l = \lambda + h.$$

The last equation gives the ratio $A : B$, the other two equations then determine A and B in terms of C. Thus C is determinable, and so is the end deflection δ, given by $C = W\lambda + P\delta$.

3. Establish the following results:

(i) $\displaystyle\int_0^\infty J_1(x)\,dx = 1.$

(ii) $\displaystyle\int_0^\infty J_n(x)\,dx = \int_0^\infty J_{n+2}(x)\,dx.$

(iii) $\displaystyle\int_0^\infty x^{-\frac{1}{2}}J_{\frac{1}{2}}(x)\,dx = (\tfrac{1}{2}\pi)^{\frac{1}{2}}.$

(iv) $\displaystyle\int_0^\infty x^{-1}J_2(x)\,dx = \tfrac{1}{2}.$

(v) $\displaystyle\int_0^\infty x^{-n}J_{n+1}(x)\,dx = \frac{1}{2^n\Gamma(n + 1)}.$

4. Discuss the following argument: By the repeated use of the recurrence formula it can be established that

$$\tfrac{1}{2}\int_0^x J_0(x)\,dx = J_1(x) + J_3(x) + J_5(x) + \cdots.$$

Hence by allowing x to tend to infinity the value of the integral is zero.

CHAPTER VII

The Modified Functions

7·1. Function with imaginary argument.

It frequently happens in physical investigations that we need Bessel functions of purely imaginary argument. There need not on that account be anything imaginary about the functions, any more than there is anything imaginary about $\cos x$, which happens to be definable by means of the imaginary exponential $\exp ix$. These functions satisfy a differential equation which is a modification of Bessel's equation different from any we have yet considered. In a more elementary field we have analogously that $\cos \omega t$ is a solution of the equation $\ddot{x} + \omega^2 x = 0$, whilst $\cosh \omega t$, or $\cos i\omega t$, is a solution of the allied equation $\ddot{x} - \omega^2 x = 0$.

We know that a solution of the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

is

$$J_n(x) = \frac{(\frac{1}{2}x)^n}{\Gamma(n+1)}\left\{1 - \frac{(\frac{1}{2}x)^2}{1(n+1)} + \frac{(\frac{1}{2}x)^4}{1\,.\,2(n+1)(n+2)} - \dots\right\}.$$

Replacing x by ix and dx by $i\,dx$, we conclude that

$$J_n(ix) = \frac{(\frac{1}{2}ix)^n}{\Gamma(n+1)}\left\{1 - \frac{(\frac{1}{2}ix)^2}{1(n+1)} + \dots\right\}$$

is a solution of the equation

$$\text{(1)} \qquad \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{n^2}{x^2}\right)y = 0,$$

and so, too, is any numerical multiple of it. Note that only one of the four terms differs in sign from Bessel's equation.

We remove the undesirable imaginary in defining a new function by the relation

$$\text{(2)} \qquad I_n(x) = (i)^{-n}J_n(ix),$$

whence

$$(3)\quad I_n(x) = \frac{(\frac{1}{2}x)^n}{\Gamma(n+1)}\left\{1 + \frac{(\frac{1}{2}x)^2}{1(n+1)} + \frac{(\frac{1}{2}x)^4}{1\,.\,2(n+1)(n+2)} + \ldots\right\}.$$

This function $I_n(x)$ is known as the modified Bessel function of the first kind, and it satisfies the equation (1). Certain of its properties strike the eye immediately (fig. 8) and show that its behaviour is quite different from that of $J_n(x)$. For positive values of x, which are the only ones that matter in practice, and for positive orders, every

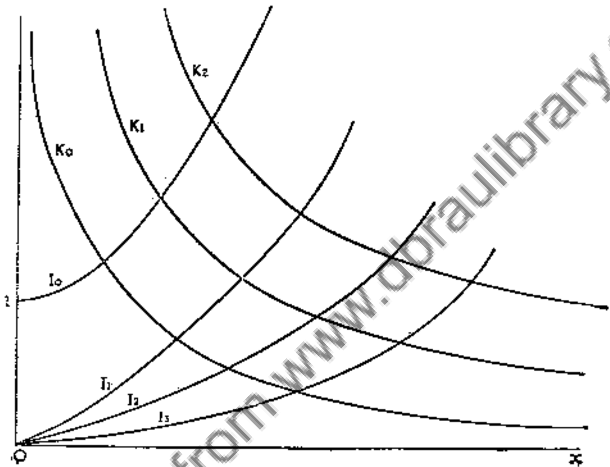

Fig. 8.—March of I_n and K_n

term in the bracket is positive and increases with x. We conclude that in these circumstances $I_n(x)$ cannot have a positive zero, and the external factor $(\frac{1}{2}x)^n$ shows that the function passes through the origin. An immediate corollary is that no solution of the equation (1) can have two zeros; for by the general theorem on the interlacing of zeros, $I_n(x)$ would then have a zero between them, and this we know is false.

The unique zero order gives

$$(4)\qquad I_0(0) = 1 = J_0(0).$$

For negative orders, the external factor $(\frac{1}{2}x)^n$ becomes infinite when x approaches zero and the function is asymptotic to the y axis; but negative orders are not particularly interesting. Just as we discarded J_{-n} in favour of a second solution Y_n, so we shall later discard the negative order function I_{-n} for a second solution K_n. For negative integral orders we write the relation $J_n = (-1)^n J_{-n}$ in the form $(i)^{-n}J_n = (i)^n J_{-n}$, which proves that

$$I_n(x) = I_{-n}(x). \tag{5}$$

Note that there is now no ambiguity of sign; the functions are identical. We shall accordingly be faced with the problem of finding an independent solution for integral orders. The series definition of the function partially breaks down when n is a negative integer. Some of the denominators become zero and the factor $\Gamma(n + 1)$ becomes infinite. The reconciliation is then the same as was employed in the parallel case of the function $J_{-n}(x)$.

7·2. Recurrence formulæ.

The Bessel functions of order $\frac{1}{2}$ are known to be expressible trigonometrically, and on switching over to an imaginary argument it is reasonable to expect the modified functions of order $\frac{1}{2}$ to be expressible in terms of hyperbolic functions. This is the case and the verification is left to the reader among the exercises. Similarly it is natural to expect recurrence formulæ, and in fact we might have investigated the functions from this end, as suggested in 3·6, Ex. 8. Replacing x by ix in the relation

$$x\frac{d}{dx}J_n(x) = nJ_n(x) - xJ_{n+1}(x),$$

we have

$$ix\frac{d}{d(ix)}J_n(ix) = nJ_n(ix) - ixJ_{n+1}(ix) = x\frac{d}{dx}J_n(ix).$$

Multiply this by $(i)^{-n}$ and it reads

$$x\frac{d}{dx}I_n(x) = nI_n(x) + xI_{n+1}(x).$$

Note the change of sign in the last term. It is then a simple matter to deduce the five relations

$$xI'_n + nI_n = xI_{n-1}, \tag{1}$$

$$xI_n' - nI_n = xI_{n+1}, \tag{2}$$

$$I_{n-1} - I_{n+1} = \frac{2n}{x}I_n, \tag{3}$$

$$I_{n-1} + I_{n+1} = 2I_n', \tag{4}$$

$$I_0' = I_1. \tag{5}$$

It has already been pointed out that for integral orders, every term in the series for I_n is positive and increases with x. The same is therefore true for I_n' and I_n''. Hence the graph rises from the origin ever more steeply, rather like the graph of $\sinh x$. The relation

$$I_{n+1} = I_{n-1} - \frac{2n}{x} I_n$$

indicates that I_{n+1} lies lower than I_{n-1} (fig. 8). For a given x, the value of the function decreases as the order rises, so that the higher orders are slower in taking off from the x axis. We can illustrate this from the tables by considering, say, $I_n(4)$ for different values of n. We find

n,	0	1	2	3	4	5	6	7	8
$I_n(4)$,	11·30	9·76	6·42	3·34	1·42	0·50	0·15	0·04	0·01

Two solutions of the same second order equation are necessarily connected, and the appropriate modification of the relation 4·4(1)

$$J_n J_{-n}' - J_n' J_{-n} = -\frac{2 \sin n\pi}{\pi x}$$

gives

(6) $$I_n I_{-n}' - I_n' I_{-n} = -\frac{2 \sin n\pi}{\pi x},$$

whence other similar results can be derived.

7·3. Standard form.

The functions that appear in the solution of various problems can hardly be expected to take the simple form $I_n(x)$ and we accordingly require a general standard for comparison. If we replace β by $i\beta$ in our previous standard 3·6(2) we conclude that $x^\alpha J_n(i\beta x^\gamma)$ or any numerical multiple of it, which includes $x^\alpha I_n(\beta x^\gamma)$, is a solution of the equation

(1) $$\frac{d^2y}{dx^2} + \frac{1-2\alpha}{x}\frac{dy}{dx} - \left[(\beta\gamma x^{\gamma-1})^2 + \frac{n^2\gamma^2 - \alpha^2}{x^2}\right]y = 0.$$

This will be taken as our standard of reference. It again appears that if $\alpha = \frac{1}{2}$ the equation is in the normal form; conversely, when the equation is in the normal form the solution must contain the factor $x^{\frac{1}{2}}$.

7·4. Lommel integrals.

We can work out the corresponding Lommel integrals by putting $\alpha = \frac{1}{2}$, $\gamma = 1$. Taking two functions of the same order, let

$$u = x^{\frac{1}{2}} I_n(\lambda x), \quad v = x^{\frac{1}{2}} I_n(\mu x).$$

The corresponding normal equations are

$$u'' - \left\{\lambda^2 + \frac{n^2 - \frac{1}{4}}{x^2}\right\} u = 0,$$

$$v'' - \left\{\mu^2 + \frac{n^2 - \frac{1}{4}}{x^2}\right\} v = 0,$$

where dashes denote differentiations with respect to x. The elimination of n leads to

$$(\lambda^2 - \mu^2)\int uv\,dx = vu' - v'u,$$

which is

(1) $$(\lambda^2 - \mu^2)\int_0^x x I_n(\lambda x) I_n(\mu x)\,dx = x\{\lambda I_n(\mu x) I_n'(\lambda x) - \mu I_n'(\mu x) I_n(\lambda x)\},$$

where the dashes now denote differentiation with respect to the argument. No suitable upper limit exists for making the right side zero; the function I_n accordingly has no orthogonal property.

If we take the two functions to be of different orders with the same argument, so that

$$u = x^{\frac{1}{2}} I_n(\lambda x), \quad v = x^{\frac{1}{2}} I_m(\lambda x),$$

we have the normal equations

$$u'' = \left\{\lambda^2 + \frac{n^2 - \frac{1}{4}}{x^2}\right\} u,$$

$$v'' = \left\{\lambda^2 + \frac{m^2 - \frac{1}{4}}{x^2}\right\} v.$$

From these we deduce

$$(n^2 - m^2)\int \frac{uv}{x^2}\,dx = u'v - uv',$$

which is

(2) $$(n^2 - m^2)\int_0^x I_n(\lambda x) I_m(\lambda x)\,\frac{dx}{x} = \lambda\{I_n'(\lambda x) I_m(\lambda x) - I_n(\lambda x) I_m'(\lambda x)\}.$$

Finally, if we multiply the equation

$$y'' + \frac{1}{x}\frac{dy}{dx} = \left(\lambda^2 + \frac{n^2}{x^2}\right)y, \quad y = I_n(\lambda x),$$

by $2x^2y'$ and integrate, we derive

$$x^2y'^2 = (\lambda^2x^2 + n^2)y^2 - 2\lambda^2\int xy^2\,dx,$$

which is

(3) $$2\int x\{I_n(\lambda x)\}^2\,dx = \left(x^2 + \frac{n^2}{\lambda^2}\right)\{I_n(\lambda x)\}^2 - x^2\{I_n'(\lambda x)\}^2.$$

EXERCISES

1. Check one of the recurrence formulæ against the following values

$$I_3 = 2{\cdot}0661 \qquad I_4 = 0{\cdot}8104 \qquad I_5 = 0{\cdot}2651 \qquad x = 3{\cdot}6.$$

2. Prove that

$$I_{\frac12}(x) = \left(\frac{2}{\pi x}\right)^{\frac12}\sinh x, \quad I_{-\frac12}(x) = \left(\frac{2}{\pi x}\right)^{\frac12}\cosh x.$$

Evaluate for the orders $\pm 1\frac12$, $\pm 2\frac12$.
Enunciate and prove the general proposition.

3. $$-\frac{2\sin n\pi}{\pi x} = I_nI_{-n+1} - I_{-n}I_{n-1}$$
$$= I_nI_{-n-1} - I_{-n}I_{n+1}.$$

4. $$\left(\frac{d}{x\,dx}\right)^r\{x^nI_n\} = x^{n-r}I_{n-r},$$

$$\left(\frac{d}{x\,dx}\right)^r\left\{\frac{I_n}{x^n}\right\} = \frac{I_{n+r}}{x^{n+r}},$$

$$\frac{d}{dx}\{x^{\frac12 n}I_n(\beta x^{\frac12})\} = \tfrac12\beta x^{\frac12(n-1)}I_{n-1}(\beta x^{\frac12}),$$

$$\frac{d}{dx}\{x^{-\frac12 n}I_n(\beta x^{\frac12})\} = \tfrac12\beta x^{-\frac12(n+1)}I_{n+1}(\beta x^{\frac12}).$$

5. Prove $I_0(x)$ is a solution of the equation $\dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) = xy.$

6. $x^2I_n'' = \{n(n-1) + x^2\}I_n - xI_{n+1}.$

7. (i) $\frac12 xI_{n-1} = nI_n + (n+2)I_{n+2} + (n+4)I_{n+4} + \ldots .$

(ii) $\frac12\int_0^x I_{n-1}(x)\,dx = I_n - I_{n+2} + I_{n+4} - \ldots .$

8. $$2^r\frac{d^r}{dx^r}I_n = I_{n-r} + rI_{n-r+2} + \frac{r(r-1)}{1\,.\,2}I_{n-r+4} + \ldots + I_{n+r}.$$

9. Prove from the recurrence formula that $I_{-n}(x)$ has no positive zero if $0 < n < 1$.

10. Obtain the expansion for $I_n(x)$ by using the method of Frobenius to solve the differential equation.

11. Show that the second solution for zero order contains the term $I_0(x)\log x$.

12. Determine whether the equation $xy'' = y$ can be solved in terms of I_n.

[Ans. $x^{\frac{1}{2}}I_1(2x^{\frac{1}{2}})$.]

13. If μ is a zero of $J_n(t)$, prove that

$$(\lambda^2 + \mu^2)\int_0^1 xI_n(\lambda x)J_n(\mu x)\,dx = \mu I_n(\lambda)J_{n+1}(\mu).$$

14. Prove that

$$\int \frac{dx}{xI_n^2(x)} = -\frac{\pi}{2\sin n\pi}\frac{I_{-n}(x)}{I_n(x)}.$$

15. In view of Ex. 2 above, prove that if the order is half an odd integer, positive or negative, the function $I_n(x)e^{-x}$ for large values of x is asymptotic to $(2\pi x)^{-\frac{1}{2}}$. An appeal to the principle of continuity makes it reasonable to suppose that the proposition holds for all orders.

16. If y satisfies the equation 7·1(1) and a new variable z is defined by $y = ze^x$, prove that z satisfies the equation

$$\frac{d^2z}{dx^2} + \left(2 + \frac{1}{x}\right)\frac{dz}{dx} + \frac{x - n^2}{x^2}z = 0.$$

7·5. The second solution.

The complete solution of the equation 7·1(1) is $y = AI_n(x) + BI_{-n}(x)$ where A and B are arbitrary constants, and no solution can have any form but this. There is effectively only one constant here when the order is integral, and in almost all research that involves Bessel functions the advent of integral orders is inevitable. The problem of finding the second independent solution for integral orders can be approached in various ways, each with its own disadvantages and compensating advantages. We could employ the method of Frobenius, with necessary modifications; alternatively we could define the function by $Y_n(ix)$; or we could use the equation

$$y = I_n(x)\int \frac{dx}{x\{I_n(x)\}^2}.$$

We fall back on a previous line of argument instead. If the equation 7·1(1) be differentiated with respect to the order n, we have

$$\frac{d^2}{dx^2}\left\{\frac{\partial I_n}{\partial n}\right\} + \frac{1}{x}\frac{d}{dx}\left\{\frac{\partial I_n}{\partial n}\right\} - \left(1 + \frac{n^2}{x^2}\right)\frac{\partial I_n}{\partial n} = \frac{2n}{x^2}I_n.$$

Similarly since I_{-n} is a solution we have

$$\frac{d^2}{dx^2}\left\{\frac{\partial I_{-n}}{\partial n}\right\} + \frac{1}{x}\frac{d}{dx}\left\{\frac{\partial I_{-n}}{\partial n}\right\} - \left(1+\frac{n^2}{x^2}\right)\frac{\partial I_{-n}}{\partial n} = \frac{2n}{x^2} I_{-n}.$$

If for brevity we put

$$K = \frac{\partial}{\partial n}(I_n - I_{-n}),$$

we have by subtraction

$$\frac{d^2K}{dx^2} + \frac{1}{x}\frac{dK}{dx} - \left(1+\frac{n^2}{x^2}\right)K = \frac{2n}{x^2}(I_n - I_{-n}).$$

When n is integral the right side is zero and we conclude that K is then a solution of equation 7·1(1). The procedure is analogous to that for obtaining Y_n from J_n.

What is required is a definition that holds for non-integral orders and which is equivalent to K for integral orders. A selection, due to various writers, is available and we may adopt

(1)
$$K_n(x) = \tfrac{1}{2}\pi \frac{I_{-n} - I_n}{\sin n\pi},$$

which is due to Macdonald. This being a linear combination of I_n and I_{-n} is certainly a solution for non-integral orders. When n is an integer it takes the indefinite form 0/0; but the usual procedure of the differential calculus gives

$$\frac{d}{dn}\sin n\pi = \pi\cos n\pi = \pi(-)^n,$$

so that
$$K_n(x) = \tfrac{1}{2}(-)^n\left\{\frac{\partial}{\partial n}(I_{-n} - I_n)\right\}.$$

This is the modified Bessel function of the second kind. It is rather a tedious business to find its explicit form for integral orders and the result is not very informative when acquired; the properties of the function are not ascertained from its series.

7·6. Recurrence formulæ.

The function naturally satisfies recurrence formulæ but they are different from any yet encountered. This is one of the disadvantages previously mentioned; whereas J_n and Y_n have the same recurrence formulæ, the modified functions I_n and K_n as accepted herein do not.

It must be borne in mind when consulting other works that slight differences may make their appearance, due to the particular definition adopted for K_n.

If we replace n by $-n$ in the definition 7·5(1), both numerator and denominator change sign, so that

(1) $$K_n(x) = K_{-n}(x).$$

The companion formula to

$$xI_n' = nI_n + xI_{n+1}$$

is

$$xI_n' = -nI_n + xI_{n-1}.$$

Changing the sign of n in the latter we have

$$xI_{-n}' = nI_{-n} + xI_{-n-1}.$$

Subtract the first equation and divide by $\sin n\pi$. In virtue of the relation $\sin(n+1)\pi = -\sin n\pi$ we have

(2) $$xK_n' = nK_n - xK_{n+1}.$$

Changing the sign of n and using (1) we have

(3) $$xK_n' = -nK_n - xK_{n-1}.$$

Subtraction then gives

(4) $$xK_{n+1} = 2nK_n + xK_{n-1},$$

whilst addition gives

(5) $$2K_n' = -K_{n+1} - K_{n-1}.$$

These have been established on the assumption that n is not integral; the justification for extending them to integral orders is an appeal to their continuity as functions of their order.

Since I_n and K_n are two independent solutions of the same reduced equation of the second order they must be related. Using the definition 7·5(1) we have

$$I_nK_n' - I_n'K_n = \tfrac{1}{2}\pi \operatorname{cosec} n\pi\{I_n(I_{-n}' - I_n') - I_n'(I_{-n} - I_n)\}$$

$$= \tfrac{1}{2}\pi \operatorname{cosec} n\pi\{I_nI_{-n}' - I_n'I_{-n}\}$$

(6) $$= \tfrac{1}{2}\pi \operatorname{cosec} n\pi\left\{-\frac{2\sin n\pi}{\pi x}\right\} = -\frac{1}{x},$$

which can be thrown into various forms by use of the recurrence formulæ.

When the order is zero we have from the recurrence formulæ

(7) $$K_0' = -K_1 = -K_{-1}.$$

When the order is $\frac{1}{2}$ we have from the definition and from the previous 7·4, Ex. 2

$$K_{\frac{1}{2}}(x) = \tfrac{1}{2}\pi \operatorname{cosec} \tfrac{1}{2}\pi\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}(\cosh x - \sinh x).$$

(8) $$= \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} = K_{-\frac{1}{2}}(x).$$

Since

$$K_{3/2} = \frac{1}{x} K_{\frac{1}{2}} + K_{-\frac{1}{2}} = \left(1 + \frac{1}{x}\right) K_{\frac{1}{2}},$$

it is readily deduced that K_n, when n is half an odd integer, is $K_{\frac{1}{2}}$ multiplied by a polynomial in x^{-1}. Consequently all such functions are asymptotic to the x axis like $x^{-\frac{1}{2}}e^{-x}$, and an appeal to the principle of continuity warrants the belief that the function behaves similarly whatever the order. This is another aspect of the asymptotic values previously mentioned in the concluding remarks of the last chapter.

7·7. Graph of K_n.

Coming now to the graph of K_n, it has to be frankly admitted that satisfactory proofs of an elementary nature are not to be had, especially of the outstanding proposition that $K_n(x)$ has no positive zeros. The graph resembles a Boyle's law curve, or the rectangular hyperbola xy = constant in the first quadrant. In default of rigidity the following is offered as a second best.

The relation 7·6(6) can be written

$$\frac{d}{dx}\left\{\frac{K_n}{I_n}\right\} = -\frac{1}{xI_n^2}.$$

The right side is essentially negative and steadily decreases in absolute value to zero as x approaches infinity. The function K_n/I_n has no stationary values; its graph ultimately becomes horizontal and the function approaches a constant value which we may denote by c. This c must be zero; for since I_n ultimately increases without limit, so too must K_n if c is anything but zero. But K_n certainly does not increase indefinitely when the order is half an odd integer, and it seems reasonable to think it does not do so in any case. Hence we take c to be zero. And as I_n is always positive, the conclusion seems to be that K_n lies wholly in the first quadrant.

Corroboration is afforded by the previous 7·4, Ex. 11. If K_0 comes down from positive infinity, we have K_0' negative and therefore K_1 positive. The recurrence formula

$$xK_2 = 2K_1 + xK_0$$

then shows that K_2 is positive, and so on. The slope is given by

$$xK_1' = -(K_1 + xK_0),$$
$$xK_2' = -(2K_2 + xK_1),$$

and so on, so that they are all negative. The relation

$$K_{n+1} = K_{n-1} + \frac{2n}{x} K_n$$

shows that, for a given x, K_{n+1} is greater than K_{n-1} and the function increases in value with the order (fig. 8). The following illustration taken almost at random from the tables shows the growth of $K_n(2·2)$ with the order:

n,	0	1	2	3	4	5	6	7	8
K_n,	0·089	0·108	0·187	0·449	1·411	5·578	26·77	151·6	991

Looking at the same functions with the doubled argument 4·4, the figures are

K_n,	0·007	0·008	0·011	0·018	0·035	0·081	0·219	0·679	2·38,

which show how very quickly the function approaches the x axis.

7·8. Lommel integrals.

The Lommel integrals concerning I can evidently be taken over bodily, merely writing K for I. There is, however, one important difference. It is no longer permissible to use the origin as the lower limit of integration. In compensation, we can use infinity as the upper limit. The fact that K_n tends to zero as x tends to infinity is one of its most valuable assets.

7·9. Zeros of the modified equation.

The modified equation has certain peculiarities that are worth noticing, though they are rarely mentioned. It has already been mentioned that no solution can have more than one zero, and a reason was assigned. If

$$y = AI_n(x) + BK_n(x)$$

is a solution, we have

$$\frac{d}{dx}\left\{\frac{y}{I_n}\right\} = B\frac{d}{dx}\left\{\frac{K_n}{I_n}\right\} = \frac{-B}{xI_n{}^2}.$$

Hence

$$\left[\frac{y}{I_n}\right]_\alpha^\beta = -B\int_\alpha^\beta \frac{dx}{xI_n{}^2}.$$

If α, β are supposed zeros of y, the left side is zero; but the right side, with its positive integrand, cannot integrate to zero, whatever the limits. Hence y cannot have more than one zero.

There is no loss of generality in taking both the arbitrary A, B to be positive. It can then be shown that a solution certainly can have one zero. We merely adopt the solution $y = AI_n - BK_n$ which ranges from negative infinity at the left to positive infinity at the right, and accordingly crosses OX. This solution is certainly inflected and cannot have a stationary value. The proof of both statements comes immediately from

$$y' = A\left\{I_{n+1} + \frac{n}{x}I_n\right\} + B\left\{K_{n-1} + \frac{n}{x}K_n\right\}.$$

The right side is essentially positive and y' cannot be zero, so that there is no stationary value. On the other hand, y' ranges from positive infinity at the left to positive infinity at the right. It must accordingly reach a minimum somewhere between, and a minimum of y' is an inflection.

In parallel with the above, the alternative form of the solution $y = AI_n + BK_n$ can have no zero, the function being always positive. Since it ranges from infinity at the left to infinity at the right, it must have a minimum somewhere between. Hence it either crosses OX at least twice or not at all. Since it cannot cross OX twice, it does not cross at all. From the differential coefficient we have

$$y' = A\left\{I_{n+1} + \frac{n}{x}I_n\right\} - B\left\{K_{n-1} + \frac{n}{x}K_n\right\}.$$

This ranges from negative infinity at the left to positive infinity at the right. It therefore has a zero, which again proves that the function has a minimum. To sum up, a solution must have either a zero or a minimum and cannot have both. No solution can touch OX except at the origin, for this would be equivalent to two positive zeros, and anyway the matter was disposed of in the theoretical work of Chap. II.

EXERCISES

1. Establish the following results:

$$\text{(i)}\quad \frac{d}{dx}\{x^n K_n(x)\} = -x^n K_{n-1}(x).$$

$$\text{(ii)}\quad \frac{d}{dx}\{x^{-n} K_n(x)\} = -x^{-n} K_{n+1}(x).$$

$$\text{(iii)}\quad K_{n+1}I_n + K_n I_{n+1} = \frac{1}{x}.$$

$$\text{(iv)}\quad \int_x^\infty \{K_{n+1}{}^2 - K_{n-1}{}^2\}x\,dx = 2nK_n{}^2(x).$$

2. Prove that

$$I_n'' = I_{n+2} + \frac{2n+1}{x} I_{n+1} + \frac{n(n-1)}{x^2} I_n,$$

$$K_n'' = K_{n-2} + \frac{2n-1}{x} K_{n-1} + \frac{n(n+1)}{x^2} K_n.$$

Deduce that, for positive integral orders, I_n'' is positive and steadily increases, whilst K_n'' is positive and steadily decreases.

3. Deduce from the last exercise that the solution which has a zero has an inflection; the solution which has no zero has no inflection.

4. If a solution has more than one stationary value, it must have an odd number of them; prove that it cannot have more than one.

5. If c, n and m are positive numbers, prove that the equation

$$I_n(x) = cK_m(x)$$

has only one solution.

6. Verify that

$$x^{\frac{1}{2}}\{AI_{n+\frac{1}{2}}(cx) + BK_{n+\frac{1}{2}}(cx)\}$$

is the solution of

$$\frac{d^2y}{dx^2} = \left[c^2 + \frac{n(n+1)}{x^2}\right]y.$$

7. Establish the following results:

$$\text{(i)}\quad \frac{d}{dx}\{I_0(cx^{\frac{1}{2}})\} = \tfrac{1}{2}cx^{-\frac{1}{2}}I_1(cx^{\frac{1}{2}}).$$

$$\text{(ii)}\quad \frac{d}{dx}\{K_0(cx^{\frac{1}{2}})\} = -\tfrac{1}{2}cx^{-\frac{1}{2}}K_1(cx^{\frac{1}{2}}).$$

$$\text{(iii)}\quad \frac{d}{dx}\{x^{\frac{1}{2}}I_1(cx^{\frac{1}{2}})\} = \tfrac{1}{2}cI_0(cx^{\frac{1}{2}}).$$

$$\text{(iv)}\quad \frac{d}{dx}\{x^{\frac{1}{2}}K_1(cx^{\frac{1}{2}})\} = -\tfrac{1}{2}cK_0(cx^{\frac{1}{2}}).$$

7·10. Motion of an augmenting mass.

Having disposed of the necessary theoretical preliminaries, we can now turn our attention to some applications of the modified Bessel functions. We begin with an example taken from dynamics.

Problem II.—*A moving body suffers accretion, so that its mass M at time t is $m(c+t)/c$. It is repelled from the origin, the force per unit mass being proportional to the distance. Initially the body is ejected from the origin with velocity u. Discuss the subsequent motion.*

Equating the force to the rate of change of linear momentum, we have the equation of motion

$$\frac{d}{dt}\left\{M\frac{dx}{dt}\right\} = Mk^2x,$$

where k^2 is the constant of proportionality. Since $M = m(c+t)/c$ we have by logarithmic differentiation

$$\frac{1}{M}\frac{dM}{dt} = \frac{1}{c+t}.$$

The equation of motion becomes

$$\frac{d^2x}{dt^2} + \frac{1}{c+t}\frac{dx}{dt} - k^2x = 0.$$

Choosing a new independent variable defined by

$$c + t = \tau, \quad dt = d\tau,$$

we have finally

$$\frac{d^2x}{d\tau^2} + \frac{1}{\tau}\frac{dx}{d\tau} - k^2x = 0.$$

This is the modified equation in almost its simplest form, and it has the solution

$$x = AI_0(k\tau) + BK_0(k\tau).$$

From this we derive the velocity

$$\frac{dx}{dt} = \frac{dx}{d\tau} = k\{AI_1(k\tau) - BK_1(k\tau)\}.$$

It remains to determine the arbitrary constants from the initial conditions

$$x = 0 = t, \quad \tau = c, \quad \frac{dx}{dt} = u.$$

We therefore have

$$0 = AI_0(kc) + BK_0(kc),$$

$$\frac{u}{k} = AI_1(kc) - BK_1(kc).$$

These solve to

$$\frac{kA}{uK_0} = \frac{-kB}{uI_0} = \frac{1}{I_1K_0 + I_0K_1}.$$

In virtue of the relation 7·9, Ex. 1 (iii)

$$I_1(kc)K_0(kc) + I_0(kc)K_1(kc) = \frac{1}{kc},$$

we derive

$$A = cuK_0(kc), \quad B = -cuI_0(kc),$$

and the solution is completely known. It remains to examine some of its consequences. Seeing that the coefficients A, B are of opposite sign, we can at once say that the solution has a zero and an inflection, but no minimum. The dynamical interpretation of this geometrical language is that there is an instant of zero displacement and an instant of zero acceleration, but no instant of zero velocity or temporary rest. We proceed to inspect the matter a little more closely. The full expression for the displacement is

$$x = cu\{K_0(kc)I_0(k\tau) - I_0(kc)K_0(k\tau)\}.$$

Regarding the right side as a function of τ, it has the obvious zero $\tau = c$, which is simply the initial condition of motion. For the velocity we have

$$\frac{dx}{dt} = kcu\{K_0(kc)I_1(k\tau) + I_0(kc)K_1(k\tau)\}.$$

This certainly has no zero since everything on the right is essentially positive; the body accordingly has no rest position. The acceleration will be zero if $AI_1' = BK_1'$, or

$$K_0(kc)I_1'(k\tau) = I_0(kc)\left\{K_0(k\tau) + \frac{1}{k\tau}K_1(k\tau)\right\}.$$

The existence of a positive root for this equation is easily demonstrated. Differentiation of the series for I_1 shows that the left side increases steadily from a non-zero constant to infinity. Both terms in the bracket on the right steadily decrease from infinity to zero. Consequently the two graphs must cross just once and there is a unique value for τ when the velocity is minimum. The body loses velocity on leaving the origin, but afterwards it accelerates continuously. It is something of a paradox that the acceleration can temporarily vanish despite the unremitting activity of the force.

EXERCISES

(The notation of the text is used throughout)

1. If the body is initially at rest at distance h from the origin, prove that its distance x at any subsequent time is given by

$$\frac{x}{chk} = K_1(kc)I_0(k\tau) + I_1(kc)K_0(k\tau).$$

2. If the body is projected towards the origin with velocity u from a distance h, prove that it fails to reach the origin if

$$uK_0(kc) < khK_1(kc).$$

In this case its nearest approach to the origin occurs at a time given by

$$\frac{I_1(k\tau)}{K_1(k\tau)} = \frac{khI_1(kc) + uI_0(kc)}{khK_1(kc) - uK_0(kc)}.$$

3. If the body is ejected from the origin with velocity u and suffers attrition in accordance with the law

$$M = m\left(\frac{c}{c+t}\right),$$

prove that its distance at any subsequent time is given by

$$x = u\tau\{K_1(kc)I_1(k\tau) - I_1(kc)K_1(k\tau)\}.$$

Deduce that the velocity increases continuously.

Prove that the momentum at any instant is

$$mcku\{K_1(kc)I_0(k\tau) + I_1(kc)K_0(k\tau)\}.$$

Verify that this agrees with the initial conditions and find the equation that determines when it is a minimum.

7·11. The variable transmission line.

Problem 12.—*Discuss the electrical transmission along a variable line.*

A somewhat different illustration is afforded by the theory of line transmission. Suppose the coefficients are resistance R, capacity C, inductance L and leakance G, all per unit length, go and return. Let $AB = dx$ define two points on the line; the distance x need not for the moment be specified and will be measured from any convenient point, not necessarily in the line. If I be the current and V be the voltage across the loop at A, we have from elementary considerations the equations

$$-\frac{\partial V}{\partial x} = RI + L\frac{\partial I}{\partial t},$$

$$-\frac{\partial I}{\partial x} = GV + C\frac{\partial V}{\partial t}.$$

In the simple case of a leaky telegraph line it is customary to put $L = 0 = C$. This gives

$$\frac{d^2V}{dx^2} = RGV,$$

which is soluble by hyperbolic functions when RG is constant.

In the less simple case we have

$$\frac{\partial^2 V}{\partial x^2} = CL\frac{\partial^2 V}{\partial t^2} + (CR + GL)\frac{\partial V}{\partial t} + RGV.$$

In virtue of Fourier analysis one need consider only sinusoidal variations. We accordingly transform to rotating vector equations on replacing V by $V\exp(i\omega t)$. This gives

$$\frac{d^2V}{dx^2} = (R + i\omega L)(G + i\omega C)V,$$

whose solution is by analogy expressible in complex hyperbolic functions of Px, where P is the complex propagation constant defined by

$$P = \{(R + i\omega L)(G + i\omega C)\}^{\frac{1}{2}}.$$

The foregoing can be seen in E. Mallett, *Telegraphy and Telephony*, and in Kennelly's work and tables of the complex hyperbolic functions.

It is accordingly evident that if we adopt a series impedance $Z = R + i\omega L$, and a shunt admittance $Y = G + i\omega C$, the analysis can be written

$$-\frac{dI}{dx} = YV; \quad -\frac{dV}{dx} = ZI.$$

It is customary in practice to boost the inductance by loading the line, and for technical reasons that need not concern us the loading is tapered. Even the capacity may vary appreciably in a single overhead line with a pronounced sag. We accordingly abandon the constancy of the coefficients and derive

$$\frac{d}{dx}\left\{\frac{1}{Z}\frac{dV}{dx}\right\} - YV = 0,$$

or,

$$\frac{d^2V}{dx^2} - \frac{1}{Z}\frac{dZ}{dx}\frac{dV}{dx} - YZV = 0.$$

It is now apparent that, formally at least, the problem is closely allied to some of our previous work, notably the one-dimensional heat conduction in a heterogeneous medium. Among the various possibilities we may take

$$Z = Z_0 x^a, \quad Y = Y_0 x^b, \quad Y_0 Z_0 = k^2.$$

These give

$$\frac{d^2V}{dx^2} - \frac{a}{x}\frac{dV}{dx} - k^2 V x^{a+b} = 0.$$

Comparison with our general standard 7·3(1) gives

$$1 - 2\alpha = -a, \quad \beta\gamma = k, \quad 2(\gamma - 1) = a + b, \quad n = \frac{\alpha}{\gamma}.$$

From these we derive

$$\alpha = \tfrac{1}{2}(1 + a), \quad \beta = \frac{2k}{2 + a + b}, \quad \gamma = 1 + \tfrac{1}{2}(a + b), \quad n = \frac{1 + a}{2 + a + b}.$$

It appears that the case $a + b = -2$ is not soluble by Bessel functions; it leads to a well-known type of equation that is soluble by exponentials. Apart from this exception we have the solution

$$V = x^\alpha\{AI_n(\beta x^\gamma) + BK_n(\beta x^\gamma)\}.$$

Various writers have investigated simple cases. In general if $a = b$ the functions are of order $\frac{1}{2}$. A particular case of this is $a = 0 = b$. The line is then uniform and we revert to the simple case of solution by hyperbolic functions, which accords with the known values of $I_{\frac{1}{2}}$ and $K_{\frac{1}{2}}$.

The case $-a = b$ was called by Ballantine the Bessel line, and the particular case $-a = 1 = b$, which has the solution

$$V = AI_0(kx) + BK_0(kx),$$

is known as the Heaviside Bessel line. If $a = 1$, $b = 0$ the series impedance tapers and the shunt admittance is constant. The reverse case has also been investigated, as has the case where $a = -1$, $b = 0$.

A modification, parallel to some of our previous work, can be made by introducing a fictitious origin otherwise by writing $Z = Z_0(x+p)/p$. Here p is a complex number such that Z_0/p is the constant complex change of series impedance per unit length. The essential difference between this and the previous case is that there is now no corresponding real origin for the series impedance. The equation for V becomes

$$\frac{d^2V}{dx^2} - \frac{1}{p+x}\frac{dV}{dx} - YZ_0V\frac{p+x}{p} = 0.$$

The substitution

$$p + x = z, \quad dx = dz,$$

produces

$$\frac{d^2V}{dz^2} - \frac{1}{z}\frac{dV}{dz} - k^2zV = 0, \quad k^2 = \frac{YZ_0}{p},$$

on the assumption that Y is constant. Comparison with our standard 7·3(1) gives

$$1 - 2\alpha = -1, \quad \beta\gamma = k, \quad 2(\gamma - 1) = 1, \quad n\gamma = \alpha,$$

whence

$$\alpha = 1, \quad \beta = \tfrac{2}{3}k, \quad \gamma = 1\tfrac{1}{2}, \quad n = \tfrac{2}{3}.$$

The solution is therefore

$$V = z\{AI_{\frac{2}{3}}(\tfrac{2}{3}kz^{3/2}) + BK_{\frac{2}{3}}(\tfrac{2}{3}kz^{3/2})\},$$

and the current can be derived from

$$I = -\frac{1}{Z}\frac{dV}{dz}.$$

It must be borne in mind that the above is little more than a formal solution. Any attempt to utilize the formulæ for purposes of computation must take cognizance of the question of phase arising from the complexity of the constant k. This technicality lies outside our scope.

Theoretically there is no reason why both the series impedance and shunt admittance should not both vary in accordance with a law of the type $(x+p)^a$, as was done with the density and specific heat in the case of one-dimensional heat conduction in a heterogeneous medium. From our point of view these can lead to nothing particularly new or interesting, and possibly they are not of sufficient practical importance to warrant investigation. Those interested may like to consult a paper by A. T. Starr on the non-uniform transmission line which appeared on p. 1052 of Vol. 20 of the *Proceedings of the Institute of Radio Engineers* in 1932.

7·12. Equilibrium of a non-uniform tie-bar.

It is well known that in the more elementary theory of beams and struts, which usually serves to exemplify differential equations with constant coefficients, the problem is generally soluble in terms of trigonometrical functions. It is very rarely pointed out that if the strut becomes a tie and is laterally loaded, the solution involves hyperbolic functions. This being so, it is hardly surprising that certain departures of the tie from uniformity lead to a solution by the modified Bessel functions. One would expect any such solution to revert to hyperbolic functions when the departure from uniformity is made indefinitely small. We propose to discuss a problem of this type.

Problem 13.—A horizontal beam of length 2λ has uniform load w per unit length. It is simply supported at the ends and there is end-pull P. The moment of inertia of section, I, at distance x from the left end is given by $I = H(x + h)/h$. Discuss the equilibrium.

It will both illustrate the method and serve as a comparison if we briefly sketch the solution when the tie is uniform. The y axis is taken positive downwards since the deflection is in that direction; the origin can be conveniently taken in the middle since the arrangement is symmetrical. The vertical reactions at the ends are each $w\lambda$, and taking moments about any point in the right half-span we have as the equation of equilibrium

$$EIy'' = Py + \tfrac{1}{2}w(x + \lambda)^2 - w\lambda(x + \lambda).$$

This evidently has a polynomial particular integral of quadratic type. We accordingly substitute

$$y = \alpha + \beta(x + \lambda) + \gamma(x + \lambda)^2.$$

On comparing coefficients we derive

$$\alpha = -\frac{w}{n^2P}, \quad \beta = \frac{w\lambda}{P}, \quad \gamma = -\frac{w}{2P}, \quad n^2 = \frac{P}{EI}.$$

The full solution can be written

$$y = A\cosh nx + B\sinh nx + \alpha + \beta(x + \lambda) + \gamma(x + \lambda)^2.$$

The two end conditions being $y = 0$, $x = \lambda, -\lambda$ we have

$$0 = A \cosh n\lambda + B \sinh n\lambda + \alpha,$$

$$0 = A \cosh n\lambda - B \sinh n\lambda + \alpha.$$

From these we derive the arbitrary constants as $B = 0$, $A = -\alpha \operatorname{sech} n\lambda$. The solution is accordingly

$$y = \frac{w}{Pn^2}\left\{\frac{\cosh nx}{\cosh n\lambda} - 1\right\} + \frac{w}{2P}(\lambda^2 - x^2).$$

Turning now to the problem in the proposed form, we have as the equation of equilibrium

$$\frac{EH}{h}(x + h + \lambda)y'' = Py + \tfrac{1}{2}w(x + \lambda)^2 - w\lambda(x + \lambda).$$

There is again a polynomial particular integral of quadratic type, and carrying out the same procedure as before we derive

$$\alpha = -\frac{w}{Pn^2}, \quad \beta = \frac{w\lambda}{P} - \frac{w}{Phn^2},$$

$$\gamma = -\frac{w}{2P}, \quad \frac{4Ph}{EH} = \nu^2 = 4hn^2.$$

The complementary function is derived from

$$\frac{d^2y}{dx^2} - \frac{Ph}{EH}\frac{y}{x + h + \lambda} = 0.$$

The substitution $x + h + \lambda = z$ suggests itself, and if we override the objection to having two forms of the independent variable in the same expression, we have the solution 7·3(1),

$$y = z^{\frac{1}{2}}\{AI_1(\nu z^{\frac{1}{2}}) + BK_1(\nu z^{\frac{1}{2}})\} + \alpha + \beta(x + \lambda) + \gamma(x + \lambda)^2.$$

With a slight reduction on using $l = h + 2\lambda$, the end conditions left and right respectively give

$$0 = h^{\frac{1}{2}}\{AI_1(\nu h^{\frac{1}{2}}) + BK_1(\nu h^{\frac{1}{2}})\} - \frac{w}{Pn^2},$$

$$0 = l^{\frac{1}{2}}\{AI_1(\nu l^{\frac{1}{2}}) + BK_1(\nu l^{\frac{1}{2}})\} - \frac{w}{Pn^2}\frac{l}{h},$$

for the determination of the arbitrary constants. They solve to

$$\frac{w/Pn^2}{h\{I_1(\nu h^{\frac{1}{2}})K_1(\nu l^{\frac{1}{2}}) - I_1(\nu l^{\frac{1}{2}})K_1(\nu h^{\frac{1}{2}})\}}$$

$$= \frac{A}{h^{\frac{1}{2}}K_1(\nu l^{\frac{1}{2}}) - l^{\frac{1}{2}}K_1(\nu h^{\frac{1}{2}})}$$

$$= \frac{B}{l^{\frac{1}{2}}I_1(\nu h^{\frac{1}{2}}) - h^{\frac{1}{2}}I_1(\nu l^{\frac{1}{2}})}.$$

It is a matter of elementary algebra, which can be left to the reader, to show that the particular integral is expressible as

$$-\frac{w}{Pn^2}\frac{x+h+\lambda}{h} + \frac{w}{2P}(\lambda^2 - x^2),$$

so that the similarity with the previous solution is quite marked. If we allow h to become indefinitely large the beam becomes uniform in section and the variform solution should go over into the uniform solution. The term $(x+h+\lambda)/h$ evidently goes over to unity, so that three of the four terms in the uniform solution are reproduced. The proof for the remaining term requires more than average manipulative skill and we shall not pursue it here. Instead, we offer a simpler illustration of the method.

Problem 14. An investigation on a variform member leads to the result

$$C = \frac{w}{n^2}\left[1 - \frac{l^{\frac{1}{2}}}{\nu h\{I_1(\nu l^{\frac{1}{2}})K_0(\nu h^{\frac{1}{2}}) + I_0(\nu h^{\frac{1}{2}})K_1(\nu l^{\frac{1}{2}})\}}\right],$$

where $$\nu^2 = 4n^2 h, \quad l = h + \lambda.$$

It is known that as h becomes indefinitely large the member tends to uniformity, in which case the solution is known to be

$$C = \frac{w}{n^2}[1 - \operatorname{sech} n\lambda].$$

It is required to reconcile the results.

We have $$l = h + \lambda = h\left(1 + \frac{\lambda}{h}\right),$$

$$l^{\frac{1}{2}} = h^{\frac{1}{2}}\left(1 + \frac{\lambda}{h}\right)^{\frac{1}{2}} \to h^{\frac{1}{2}}\left(1 + \frac{\lambda}{2h}\right),$$

$$\nu(l^{\frac{1}{2}} - h^{\frac{1}{2}}) \to \frac{\nu\lambda}{2h^{\frac{1}{2}}} = n\lambda.$$

Testing our belief in our unproven asymptotic formulæ 7·4, Ex. 15, and 7·6(8)

$$I_n(x) \to \frac{e^x}{(2\pi x)^{\frac{1}{2}}}; \quad K_n(x) \to \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x},$$

we have

$$I_1(\nu l^{\frac{1}{2}})K_0(\nu h^{\frac{1}{2}}) \to \frac{1}{2\nu h^{\frac{1}{2}}} \exp\{\nu(l^{\frac{1}{2}} - h^{\frac{1}{2}})\} = \frac{\exp n\lambda}{2\nu h^{\frac{1}{2}}}.$$

Similarly

$$I_0(\nu h^{\frac{1}{2}})K_1(\nu l^{\frac{1}{2}}) \to \frac{\exp(-n\lambda)}{2\nu h^{\frac{1}{2}}}.$$

Hence

$$\nu h\{\quad\} \to h^{\frac{1}{2}} \cosh n\lambda,$$

and the concordance is established.

These problems on variable bars furnish physical evidence that the asymptotic formulæ are independent of the order. By putting

$$I = H\left(\frac{x+h}{h}\right)^r$$

we can, by a proper choice of r, make the resulting functions to be of any order we like. Whatever the order may be, the bar becomes uniform as h increases indefinitely. And there is only one solution of a particular problem for a uniform bar.

EXERCISES

1. In the problem of the uniformly loaded beam, prove that of the three denominators under A, B, &c., none can be zero. This proves that no adjustment of the constants will make the solution depend on the one type of function without the other. This is not usually true of solutions that depend on J_n and Y_n.

Prove further that the denominator under B is essentially negative.

2. A cantilever of length λ has end load W and end pull P. The moment of inertia of section being given by $I = H(x+h)/h$, prove that, with the notation of the text, the retaining couple C at the wall is given by

$$\nu C\{I_1(\nu l^{\frac{1}{2}})K_0(\nu h^{\frac{1}{2}}) + I_0(\nu h^{\frac{1}{2}})K_1(\nu l^{\frac{1}{2}})\}$$
$$= 2Wh^{\frac{1}{2}}\{I_1(\nu l^{\frac{1}{2}})K_1(\nu h^{\frac{1}{2}}) - I_1(\nu h^{\frac{1}{2}})K_1(\nu l^{\frac{1}{2}})\}.$$

It is known that, when the section is uniform, the corresponding solution is $C = (W/n)\tanh n\lambda$. Reconcile the two results, the latter being the limiting form as h increases indefinitely.

3. A cantilever of length λ has uniform load w per unit length and end pull

P. The retaining couple is C and the moment of inertia of section at distance x from the wall is $I = H(x+h)/h$. Verify the following details of the analysis. The equilibrium equation is

$$EH\frac{x+h}{h}y'' = C + Py + \tfrac{1}{2}wx^2 - w\lambda x.$$

The assumption of a particular integral of the form $y = \alpha + \beta x + \gamma x^2$ leads to

$$\alpha = -\frac{1}{P}\left(C + \frac{w}{n^2}\right); \quad \beta = \frac{w}{P}\left(\lambda - \frac{1}{hn^2}\right); \quad \gamma = -\frac{w}{2P}; \quad n^2 = \frac{P}{EH}.$$

The complementary function is

$$y = z^{\frac{1}{2}}\{AI_1(\nu z^{\frac{1}{2}}) + BK_1(\nu z^{\frac{1}{2}})\}, \quad \nu^2 = 4n^2h, \quad z = x + h,$$

and the end conditions at the left lead to

$$-A = 2\beta h^{\frac{1}{2}}K_1(\nu h^{\frac{1}{2}}) + \alpha\nu K_0(\nu h^{\frac{1}{2}}),$$
$$B = 2\beta h^{\frac{1}{2}}I_1(\nu h^{\frac{1}{2}}) - \alpha\nu I_0(\nu h^{\frac{1}{2}}).$$

The right end being free from applied couple, we deduce

$$\frac{Cn^2}{w} = \frac{[\quad]}{2nh\{\quad\}} - 1, \quad l = h + \lambda,$$

where

$$[\quad] = \left(\frac{l}{h}\right)^{\frac{1}{2}} + 2(1 - n^2h\lambda)[K_1(\nu l^{\frac{1}{2}})I_1(\nu h^{\frac{1}{2}}) - K_1(\nu h^{\frac{1}{2}})I_1(\nu l^{\frac{1}{2}})],$$
$$\{\quad\} = I_1(\nu l^{\frac{1}{2}})K_0(\nu h^{\frac{1}{2}}) + I_0(\nu h^{\frac{1}{2}})K_1(\nu l^{\frac{1}{2}}).$$

It can be shown that, in the case of a uniform beam, the corresponding solution is

$$\frac{Cn^2}{w} = \frac{1 + n\lambda\sinh n\lambda}{\cosh n\lambda} - 1.$$

CHAPTER VIII

Applications to Hydrodynamics and Elasticity

8·1. Tidal motion.

The problems so far discussed in illustration of the theory have been of a relatively simple kind, involving but little analysis. We come now to a number of problems of somewhat more complicated type, beginning with surface waves on a liquid. These are distinct from waves of expansion which are propagated throughout the body of a medium, like waves of sound or light; and a further distinction is drawn according as the liquid is relatively shallow or deep. We begin with the former.

In a current of liquid, the motion of the particles is secular and the liquid continuously moves forward with the current. The contrast in wave motion is the presumption that a particle makes only a limited excursion from a mean position, its displacement being a compound of vertical and horizontal oscillations.

It is first necessary to set up two fundamental equations. One of these is a physical equation, known as the equation of continuity and based on the incompressibility of the liquid. It states that the mass of liquid in any specified space-element is fixed by the volume of the element and is invariable. The second equation is purely dynamical.

Consider a horizontal canal running in the direction OX. Any surface phenomenon will be presumed to be uniform across the breadth of the canal, so we are working with two dimensions in a vertical plane. The x axis is in the surface of the liquid in its rest position and we take two fixed vertical planes L, M defined by x and $x + \delta x$. The surface breadth in L being b, we take h to be the mean depth of the possibly non-uniform cross-section A, so that $A = bh$.

When the surface is agitated by a wave we require two moving co-ordinates. We take η to be the small surface elevation above the rest position, uniform across the breadth in plane L; ξ to be the horizontal forward displacement of any particle in the plane L, so that the velocity $\partial\xi/\partial t$ is uniform over the plane L and extends from the surface to the canal bed. The justification for this assumption appears in the consistency of the results.

The rate of liquid entry into the lamina LM from the left is $Q = bh\partial\xi/\partial t$. The corresponding exit across M is $Q + \delta Q$. The accession $-\delta Q$ in the lamellar space between L and M causes a surface rise $\partial\eta/\partial t$ over an area measuring b by δx. This gives

$$b\,\frac{\partial\eta}{\partial t}\,\delta x = -\frac{\partial}{\partial x}\left\{bh\,\frac{\partial\xi}{\partial t}\right\}\delta x,$$

equivalent to

$$b\eta + \frac{\partial}{\partial x}(bh\xi) = 0,$$

as the equation of continuity.

Coming now to the dynamical equation, consider an element of length δx and small cross-section a, so that its mass is $\rho a\delta x$ and moves with an acceleration $\partial^2\xi/\partial t^2$. This acceleration is caused by the excess pressure at the one end as compared with the other, and this in turn is due to the difference of head at the planes L and M. We thus have

$$\rho a\,\frac{\partial^2\xi}{\partial t^2}\,\delta x = -\rho g a\,\frac{\partial\eta}{\partial x}\,\delta x,$$

or,

$$\frac{\partial^2\xi}{\partial t^2} = -g\,\frac{\partial\eta}{\partial x}.$$

We can now eliminate ξ and derive

$$\frac{g}{b}\,\frac{\partial}{\partial x}\left\{bh\,\frac{\partial\eta}{\partial x}\right\} = \frac{\partial^2\eta}{\partial t^2}.$$

The particular case of a uniform canal, with b and h both constant, is not our immediate concern. It evidently leads to

$$gh\,\frac{\partial^2\eta}{\partial x^2} = \frac{\partial^2\eta}{\partial t^2},$$

and the solution is known to be waves travelling with velocity $(gh)^{\frac{1}{2}}$ and possibly giving standing waves. Leaving this aside and coming to canals of variable section, we assume that η has a periodic motion of small amplitude and proportional to $\cos(\omega t + \epsilon)$. We then have

$$\frac{d}{dx}\left\{bh\,\frac{d\eta}{dx}\right\} + \frac{b\omega^2}{g}\,\eta = 0.$$

Much of hydrodynamics is so difficult that it is common practice first to solve the necessary equations and afterwards seek the problem which the solution fits. One of the simplest hypotheses that one can make in the above equation is to treat h as constant and take b proportional to x. This gives a crude representation of a V-shaped estuary, of uniform depth, that fans out from the origin. The equation becomes

$$\frac{d^2\eta}{dx^2} + \frac{1}{x}\frac{d\eta}{dx} + k^2\eta = 0, \quad k^2 = \frac{\omega^2}{gh}.$$

The solution which gives a finite elevation at the origin is $\eta = BJ_0(kx)$, and if we suppose that the distance to the mouth is λ where the open sea has a tidal rise given by $\eta = C\cos(\omega t + \epsilon)$, the solution

$$\eta = \frac{J_0(kx)}{J_0(k\lambda)}\, C\cos(\omega t + \epsilon)$$

fits at both ends. In view of the known steady decline in the absolute value of the maxima and minima as we move to the right, we get an intimation of the increase in wave-height as the tide comes up the channel.

8·2. Canal of finite length.

The above problem is deceptively simple and there are very few like it. We turn instead to the problem of the canal of finite length. Reverting to the equation for the uniform canal, we have

$$c^2\frac{\partial^2\xi}{\partial x^2} = \frac{\partial^2\xi}{\partial t^2}, \quad c^2 = gh,$$

showing that the velocity of propagation is "due to a fall through half the depth". An infinity of solutions can be built up of the form

$$\xi = \frac{\sin}{\cos} mx \frac{\sin}{\cos} mct,$$

but if the canal is closed by two vertical barriers at $x = 0$ and $x = \lambda$ the fluid can have no horizontal velocity there. We accordingly have as boundary conditions

$$\frac{\partial\xi}{\partial t} = 0, \quad x = 0,\ \lambda.$$

A suitable solution is then $\xi = B\sin mx \sin(mct + \epsilon)$, provided

$$\sin m\lambda = 0, \quad m\lambda = \pi,\ 2\pi,\ 3\pi,\ \ldots\ .$$

This gives the possible frequencies of the normal modes of vibration, where every surface-particle goes through its equilibrium position at the same instant.

It is natural to inquire what is the analogue for a non-uniform canal. If a simple solution is desired, trouble arises from the equation of continuity. Suppose we solve the equation of motion for η; then an integration is needed to find ξ, which must be known in order to satisfy the boundary conditions. Very few expressions involving Bessel functions are readily integrable, a difficulty similar to that encountered when attempting to evaluate Fourier coefficients. Alternatively, if we solve for ξ, we have to differentiate to find η and thereby lose the simplicity of the wave form.

Taking the equation for η we assume with some generality that $bh \propto x^p$ and $b \propto x^q$. The equation becomes

$$\frac{d^2\eta}{dx^2} + \frac{p}{x}\frac{d\eta}{dx} + \eta\,\frac{\omega^2}{g}\,\sigma x^{q-p} = 0,$$

where σ is a constant.

This, of course, is readily soluble; but our chance of finding an integrable result is based on the relation

$$\frac{d}{dt}\left\{t^{\gamma(n+1)}J_{n+1}(\beta t^{\gamma})\right\} = \beta\gamma t^{\gamma(n+2)-1}J_n(\beta t^{\gamma}).$$

Comparison with our standard 3·6(2) gives

$$1 - 2\alpha = p, \quad 2\gamma - 2 = q - p, \quad \pm n = \frac{\alpha}{\gamma}.$$

If

$$\eta = x^{\alpha}J_n(\beta x^{\gamma}), \quad b\eta \propto x^{\alpha+q}J_n(\beta x^{\gamma}),$$

the integrability of

$$bh\xi = -\int b\eta\, dx$$

is ensured by taking

$$\alpha + q = \gamma(n+2) - 1, \quad q = 2\gamma - 1 \neq -1,$$

$$p = 1, \quad \alpha = 0 = n.$$

As an illustration we may take $q = 2$. The problem and its solution then reads: If the variable breadth, depth and cross-section are given by

$$b = B\left\{\frac{f+x}{f}\right\}^2, \quad h = \frac{A}{B}\frac{f}{f+x}, \quad bh = A\frac{f+x}{f},$$

the equation for η is

$$\frac{d}{dx}\left\{A\frac{f+x}{f}\frac{d\eta}{dx}\right\}+\frac{\omega^2}{g}B\left\{\frac{f+x}{f}\right\}^2\eta=0.$$

The substitution $z=f+x$ gives

$$\frac{d}{dz}\left\{z\frac{d\eta}{dz}\right\}+\frac{\omega^2B}{gfA}z^2\eta=0,$$

or,

$$\frac{d^2\eta}{dz^2}+\frac{1}{z}\frac{d\eta}{dz}+\frac{9}{4}\beta^2z\eta=0, \qquad \beta^2=\frac{4}{9}\frac{\omega^2B}{gfA}.$$

The solution is verified in the usual way to be

$$\eta=LJ_0(\beta z^{3/2})+MY_0(\beta z^{3/2}),$$

whence we derive

$$-\frac{Afz}{B}\xi=\int z^2\eta\,dz$$

$$=\frac{2}{3\beta}z^{3/2}\{LJ_1(\beta z^{3/2})+MY_1(\beta z^{3/2})\}.$$

If the ends of the canal are defined by $x=0$, $z=f$ and $x=\lambda$, $z=f+\lambda=l$, both ends are velocity nodes if

$$0=LJ_1(\beta f^{3/2})+MY_1(\beta f^{3/2}),$$
$$0=LJ_1(\beta l^{3/2})+MY_1(\beta l^{3/2}).$$

The elimination of the constants gives

$$\frac{J_1(\beta f^{3/2})}{Y_1(\beta f^{3/2})}=\frac{J_1(\beta l^{3/2})}{Y_1(\beta l^{3/2})}.$$

This transcendental equation determines β and thence ω and the normal modes of vibration.

Translated into figures, we find from the tables that $J_1/Y_1=0{\cdot}954$ when the argument is 12·56 and again when the argument is 15·71. Thus

$$\frac{\beta l^{3/2}}{\beta f^{3/2}}=\frac{15{\cdot}71}{12{\cdot}56},$$

whence we derive $l/f=1{\cdot}161$ and $f=6{\cdot}2\lambda$. This means that the fictitious origin is distant rather more than six times the length of the canal; and the variation in mean depth and cross-section is about sixteen per cent as between one end and the other. The period is calculated from

$$\beta^2 = \frac{4}{9}\frac{\omega^2 B}{gfA} = \frac{(12\cdot 56)^2}{f^3}.$$

Hence for a model tank 1 metre long, the water being 10 cm. deep at the deep end, we have $\omega = 9\cdot 48$ and the period $2\pi/\omega = 0\cdot 66$ secs.

The position of the surface-node is given by $\eta = 0$, so that

$$J_0(\beta z^{3/2}) : Y_0(\beta z^{3/2}) = 0\cdot 954.$$

Interpolating from the tables we have $\beta z^{3/2} = 14\cdot 17$. As $\beta f^{3/2} = 12\cdot 56$ we get

$$x = z - f = 52 \text{ cm}.$$

This is about what one might expect.

A good deal more latitude of choice is attained on approaching the problem through ξ. The elimination of η from the dynamical equation and the equation of continuity gives

$$\frac{d}{dx}\left\{\frac{1}{b}\frac{d}{dx}(bh\xi)\right\} + \frac{\omega^2}{g}\xi = 0.$$

If we make the same tentative assumptions regarding the breadth and area, we have, where σ is a constant,

$$\frac{d^2\xi}{dx^2} + \frac{2p-q}{x}\frac{d\xi}{dx} + \left\{\frac{p(p-q-1)}{x^2} + \frac{\omega^2}{g}\sigma x^{q-p}\right\}\xi = 0.$$

Provided $p - q \neq 2$ this is soluble in terms of Bessel functions and comparison with the standard gives 3·6(2)

$$1 - 2\alpha = 2p - q, \quad \alpha^2 - n^2\gamma^2 = p(p-q-1), \quad 2\gamma - 2 = q - p.$$

Since

$$bh\xi \propto x^{\alpha+p} J_n(\beta x^\gamma)$$

a comparatively manageable form is obtained by making $\alpha + p = \pm n\gamma$. We then have

$$n^2\gamma^2 = (\alpha + p)^2,$$

$$n^2\gamma^2 - \alpha^2 = 2p\alpha + p^2 = -p(p-q-1),$$

$$p(2\alpha + 2p - q - 1) = 0.$$

It follows that either p is zero or else the first comparison equation is satisfied automatically. The former alternative gives

$$a = \tfrac{1}{2}(1+q), \quad \gamma = 1 + \tfrac{1}{2}q, \quad n = \frac{1+q}{2+q}.$$

As a particular case of this we can take q to be also zero. The solution is then in terms of $x^{\frac{1}{2}}J_{\frac{1}{2}}(x)$, which is trigonometrical and correct since the canal is then uniform.

The second alternative gives

$$a = \tfrac{1}{2}(1+q) - p, \quad \gamma = 1 - \tfrac{1}{2}(p-q), \quad n = \frac{1+q}{2-p+q}.$$

This opens up endless possibilities and we may illustrate by taking $p = 3$, $q = 2$. The problem and its solution then read: If the variable breadth, depth and cross-section are given by

$$b = B\left\{\frac{f+x}{f}\right\}^2, \quad h = \frac{A}{B}\frac{f+x}{f}, \quad bh = A\left\{\frac{f+x}{f}\right\}^3,$$

the equation for ξ is

$$\frac{d}{dx}\left[\frac{f^2}{B(f+x)^2}\frac{d}{dx}\left\{A\frac{(f+x)^3}{f^3}\xi\right\}\right] + \frac{\omega^2}{g}\xi = 0,$$

or,

$$\frac{d}{dz}\left\{\frac{1}{z^2}\frac{d}{dz}(z^3\xi)\right\} + \frac{Bf\omega^2}{Ag}\xi = 0,$$

so that

$$\frac{d^2\xi}{dz^2} + \frac{4}{z}\frac{d\xi}{dz} + \frac{\beta^2}{4}z\xi = 0, \quad \beta^2 = \frac{4Bf\omega^2}{Ag}.$$

The solution is

$$\xi = z^{-3/2}\{LJ_3(\beta z^{\frac{1}{2}}) + MY_3(\beta z^{\frac{1}{2}})\},$$

and correspondingly

$$\eta = -\frac{\beta A}{2fB}z^{-1}\{LJ_2(\beta z^{\frac{1}{2}}) + MY_2(\beta z^{\frac{1}{2}})\}.$$

As an illustration of the difficulties likely to be encountered in translating this into figures, suppose the mean depth shows a 5 per cent variation as between the two ends. Then the breadth varies about 10 per cent and the cross-section varies about 15 per cent. This may legitimately be described as gradual. Suppose c is a zero of the cylinder function that defines ξ; then the next consecutive zero is about $c + \pi$. The end conditions therefore give

$$\frac{\beta l^{\frac{1}{2}}}{\beta f^{\frac{1}{2}}} = \frac{c + \pi}{c}.$$

Since we are assuming that $l/f = 21/20$ we have $(l/f)^{\frac{1}{2}} = 41/40$ approximately, and c is about 40π. This would exhaust the possibilities of any known tables.

It is not impossible for the solution to come out in terms of the one type of function without the other; compare 7·12, Ex. 1. For example, the thirty-fifth zero of Y_3 is 112·273, and if this happens to be the value of $\beta f^{\frac{1}{2}}$ the first equation of condition would give $L = 0$. The next zero would give $\beta l^{\frac{1}{2}} = 115{\cdot}416$, whence we conclude that f is about $17{\cdot}7\lambda$. If λ is known, so is β; the value of ω then depends on the adopted value of A/B.

8·3. Surface waves.

The type of motion hitherto considered, where the horizontal disturbance is presumed uniform throughout the depth of the liquid, and the vertical motion of a particle is more or less ignored, is known as tidal. We propose now to take a less restrictive hypothesis and to work in cylindrical co-ordinates. This will have the advantage of introducing Laplace's equation, which plays so large a part in classical mathematical physics.

Let u, v, w be the component velocities of a particle in the directions of increasing r, θ, z respectively and consider a volume element defined by the coaxial cylinders of radii r, $r + \delta r$; by horizontal planes at levels z, $z + \delta z$; by axial planes of azimuthal angles θ, $\theta + \delta\theta$.

The rate of liquid influx across the inner vertical face is

$$Q_1 = ur\,\delta\theta\,\delta z.$$

The corresponding efflux across the opposite face is $Q_1 + \delta Q_1$, and the resulting efflux from the volume element is

$$\delta Q_1 = \frac{\partial}{\partial r}(ur\,\delta\theta\,\delta z)\,\delta r.$$

Similarly from the top and bottom faces we have

$$\delta Q_2 = \frac{\partial}{\partial z}(wr\,\delta\theta\,\delta r)\,\delta z.$$

For the remaining pair of vertical faces we have

$$\delta Q_3 = \frac{\partial}{\partial \theta}(v\,\delta r\,\delta z)\,\delta\theta.$$

The incompressibility of the liquid thus gives the equation of continuity

$$\frac{\partial}{\partial r}(ur) + \frac{\partial v}{\partial \theta} + r\frac{\partial w}{\partial z} = 0.$$

It is common practice in dynamics and electrostatics to adopt an energy potential whose negative gradient gives the force in any particular direction. A similar idea is adopted in hydrodynamics, a velocity potential being assumed whose negative gradient gives the velocity in any direction. It corresponds to a type of motion known as irrotational, in contradistinction to rotational or vortex motion.

Denoting this velocity potential by ϕ, we have by definition

$$u = -\frac{\partial \phi}{\partial r}, \quad v = -\frac{\partial \phi}{r\partial \theta}, \quad w = -\frac{\partial \phi}{\partial z}.$$

The equation of continuity becomes

$$\frac{\partial}{\partial r}\left\{r\frac{\partial \phi}{\partial r}\right\} + \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta^2} + r\frac{\partial^2 \phi}{\partial z^2} = 0,$$

and this is equivalent to Laplace's equation in cylindrical co-ordinates.

The only satisfactory way of attempting to solve a partial differential equation of this type is to assume initially that ϕ has the form FH, where F is a function of r and z, whilst H is a function of θ alone. The equation can then be written

$$r\frac{\partial}{\partial r}\left\{r\frac{\partial F}{\partial r}\right\} + \frac{1}{H}\frac{\partial^2 H}{\partial \theta^2} + r^2\frac{\partial^2 F}{\partial z^2} = 0.$$

Since θ is a variable independent of r and z, this implies that the middle term must be a constant, and with a view to Fourier analysis we put

$$\frac{1}{H}\frac{dH}{d\theta} = -n^2,$$

so that $H \propto \cos(n\theta + \epsilon)$, with n an integer.

The equation is now

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial F}{\partial r}\right\} + \frac{\partial^2 F}{\partial z^2} - \frac{n^2}{r^2}F = 0.$$

There is already an adumbration of Bessel's equation, and if we put $F = RZ$, where R is a function of r alone, and Z a function of z alone, we have

$$\frac{1}{R}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r^2}R\right\} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0.$$

As before, we conclude that the last term must be a constant, and in our present case it is convenient to take $Z \propto \cosh \kappa(z + h)$. This leaves

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{dR}{dr}\right\} + \left(\kappa^2 - \frac{n^2}{r^2}\right)R = 0,$$

with the solution

$$R = AJ_n(\kappa r) + BY_n(\kappa r).$$

We now assume that the horizontal plane defined by $z = 0$ is the surface of still liquid of uniform depth h. When the water is mildly agitated, a particle at the bottom can have no vertical velocity, so that

$$w = -\frac{\partial\phi}{\partial z} = 0 \quad \text{when} \quad z = -h.$$

This condition is already fulfilled by our choice of Z.

There is nothing in the analysis so far that specifically connects it with wave motion. The connexion is made by introducing a periodic time factor into the velocity potential, which thus becomes

$$\phi = \{AJ_n(\kappa r) + BY_n(\kappa r)\}\cos(n\theta + \epsilon)\cosh\kappa(z + h)\cos mt.$$

If η denote the slight surface elevation at any point, its vertical velocity is given by

$$\dot{\eta} = -\frac{\partial\phi}{\partial z} \quad \text{when} \quad z = 0.$$

As this is proportional to $\cos mt$, integration gives $\eta \propto \sin mt$, so that the origin of time is taken when η is zero and the surface-particles are moving through their equilibrium positions.

It remains to connect the various parameters. It is shown in texts on hydrodynamics that in the absence of viscosity and impressed forces other than gravity, the plane $z = 0$ is a free surface provided

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0.$$

This evidently leads to

$$g\kappa \tanh \kappa h = m^2,$$

where incidentally $\tanh \kappa h$ is necessarily less than unity and is asymptotic to it for large values of κh.

If the liquid is confined between two retaining walls defined by $r = a, b$, the horizontal velocity u must be zero at each of these, so that

$$u = -\frac{\partial \phi}{\partial r} = 0 \quad \text{when} \quad r = a, b.$$

The arbitrary constants A, B can then be eliminated and we have

$$\frac{J_n'(\kappa a)}{Y_n'(\kappa a)} = \frac{J_n'(\kappa b)}{Y_n'(\kappa b)}$$

as the equation to determine κ.

For the sake of arriving at figures we suppose firstly that there is no inner bastion. We then discard Y_n as giving an infinity at the origin. Next, suppose the water is divided quadrantally by two vertical and perpendicular barriers. At each of these we need no normal velocity, so that

$$v = -\frac{\partial \phi}{r\partial \theta} = 0.$$

This is achieved by dropping the phase angle ϵ and putting $n = 2$. We now have

$$\phi = AJ_2(\kappa r) \cos 2\theta \cosh \kappa(z + h) \cos mt.$$

If the outer boundary is defined by $r = a$, the radial velocity must there be zero, and

$$u = -\frac{\partial \phi}{\partial r} = 0, \quad J_2'(\kappa a) = 0.$$

The recurrence relation

$$2J_n' = J_{n-1} - J_{n+1}$$

then tells us that the required zeros are to be found from

$$J_1(\kappa a) = J_3(\kappa a).$$

To find, say, the first root of this equation, J_3 is the slower in taking off from the axis and its first zero is beyond the first zero of J_1, so the required root is rather less than the first zero of J_1. From the extracted values

x	J_1	J_3
3·0	0·33905	0·30906
3·1	0·30092	0·32644

we deduce that the required zero is $3{\cdot}054 = \kappa a$.

For a tank whose depth equals its diameter we have $\kappa h = 6{\cdot}1$ and the distinction between $\tanh \kappa h$ and unity is not worth making. If the diameter is a metre, we have

$$m = (g\kappa)^{\frac{1}{2}} = 7{\cdot}72,$$

and the period of oscillation is

$$\frac{2\pi}{m} = 0{\cdot}81 \text{ sec.}$$

EXERCISES

1. Discuss the tidal waves in a channel of uniform breadth, the depth shelving uniformly from zero down to the open sea where there is a tidal rise $\eta = C\cos(\omega t + \epsilon)$. Prove that the function is of order zero and the argument proportional to $x^{\frac{1}{2}}$.

2. Discuss the normal modes of vibration for surface waves in a semicircular tank of radius a and depth h. Compute from the values

x	J_0	J_2
1·8	0·3400	0·3061
1·9	0·2818	0·3300

8·4. Heat conduction.

The analysis for heat conduction in cylindrical co-ordinates is so close to our previous work as almost to amount to a transcript. We merely call ϕ the temperature instead of the velocity potential, and pay attention to a few necessary coefficients. Taking the same cylindrical space element as before, we let H_1 be the heat per second that flows across the inner vertical face. Then $H_1 + \delta H_1$ flows across the opposite face, and the accumulation in the volume element is $-\delta H_1$. The amount H_1 depends jointly on the area which it crosses, the conductivity κ, and the negative temperature gradient radially. We thus have

$$H_1 = -\frac{\partial\phi}{\partial r}\kappa r\,\delta\theta\,\delta z, \quad -\delta H_1 = -\frac{\partial H_1}{\partial r}\delta r = +\kappa\frac{\partial}{\partial r}\left\{r\frac{\partial\phi}{\partial r}\right\}\delta\theta\,\delta z.$$

Similarly if H_2 flows upwards through the lower face we have

$$-\delta H_2 = -\frac{\partial H_2}{\partial z}\partial z = \kappa r\frac{\partial^2\phi}{\partial z^2}\delta r\,\delta\theta\,\delta z.$$

Laterally we have

$$-\delta H_3 = -\frac{\partial H_3}{\partial\theta}\delta\theta = \frac{\kappa}{r}\frac{\partial^2\phi}{\partial\theta^2}\delta r\,\delta\theta\,\delta z.$$

The total heat-gain shows itself in a rise of temperature. Taking density ρ and specific heat s, we have the heat gain per second as

$$\rho s\frac{\partial\phi}{\partial t}r\,\delta r\,\delta\theta\,\delta z.$$

The equation for ϕ is therefore

$$(1) \qquad \frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial\phi}{\partial r}\right\} + \frac{\partial^2\phi}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{\rho s}{\kappa}\frac{\partial\phi}{\partial t}.$$

The theory of heat conduction is an extensive study in itself and as this is not the place to pursue it, all we can do is to indicate how Bessel functions occasionally play a part in the analysis. When the temperature has ceased to vary with the time, we have what is known as the "steady state" and the right side of the last equation falls out, thus simplifying the working.

Problem 15.—*Hot gases are conveyed through a long straight pipe which is indifferently lagged; it is required to find the temperature distribution throughout the lagging, whose internal and external radii are d and c respectively.*

A definite length is under observation and a sufficient number of thermo-couples enables the steady temperature distribution at the outer surface to be expressed as a Fourier series. It can be assumed that the temperature is constant round the circular cross-section of any coaxial cylindrical surface. The temperature of the surrounding medium can be assumed uniform and will serve as the origin of the temperature scale. It is assumed that the heat-loss into the surrounding medium is to be accounted for by a constant coefficient of emissivity h.

The particular type of Fourier series adopted is a matter for careful discrimination since it is very easy unwittingly to introduce extraneous hypotheses which are not justifiable. The mean temperature of the outer surface is hardly likely to coincide exactly with that of the surrounding medium, so there is pretty certainly a constant term present. We accordingly suppose that the temperature of the outer surface is given by

$$\phi_e = a_0 + a_1 \cos vz + a_2 \cos 2vz + \dots$$
$$+ b_1 \sin vz + b_2 \sin 2vz + \dots .$$

The number of terms employed has to be justified by the number of thermo-couples; but in any case the coefficients are uniquely known from the observational data.

Under the given assumptions the equation for ϕ becomes

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

If we suppose the solution is of the form $\phi = R \exp i\omega z$, where R is a function of r alone, we have

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \omega^2 R = 0.$$

This puts us in possession of solutions of the type

$$\phi = \{AI_0(\omega r) + BK_0(\omega r)\} \cos \omega z + \{CI_0(\omega r) + DK_0(\omega r)\} \sin \omega z.$$

Further examination shows that these in themselves are not enough to satisfy our requirements and, as often happens, advantage has to be taken of the solution

$$\phi = p + q \log r,$$

where p and q are constants. The existence of this solution is readily demonstrated, and by taking various values for ω we can build up the solution

$$\phi = \Sigma\{A_n I_0(\omega_n r) + B_n K_0(\omega_n r)\} \cos \omega_n z$$
$$+ \Sigma\{C_n I_0(\omega_n r) + D_n K_0(\omega_n r)\} \sin \omega_n z$$
$$+ p + q \log r.$$

At the outer boundary, this solution must coincide with ψ_c. We conclude by comparison, firstly that all the values of ω are given by

$$\omega_n = nv;$$

secondly, that the independent terms are related by

$$a_0 = p + q \log c;$$

thirdly, that the other coefficients are connected by the relations

$$A_n I_0(nvc) + B_n K_0(nvc) = a_n,$$
$$C_n I_0(nvc) + D_n K_0(nvc) = b_n.$$

Each of the last three relations leaves us with effectively one unknown.

Coming now to the heat emission, we equate the heat emitted by any surface element to the heat which it receives. This gives the relation

$$-\kappa \frac{\partial \phi}{\partial r} = h\phi, \quad r = c.$$

The application of this to each term of our series gives

$$-\frac{\kappa q}{c} = h(p + q \log c) = ha_0.$$

$$B_n K_1(nvc) - A_n I_1(nvc) = \frac{ha_n}{\kappa nv},$$

$$D_n K_1(nvc) - C_n I_1(nvc) = \frac{hb_n}{\kappa nv}.$$

Taking advantage of the relation 7·9, Ex. 1 (iii)

$$I_0 K_1 + I_1 K_0 = \frac{1}{nvc},$$

all the constants are now determinable by elementary algebra and the solution may be considered complete.

EXERCISES

1. Show that the heat per second that crosses the section defined by $z = 0$ can be expressed as

$$\frac{2\pi ch}{v} \Sigma \frac{b_n}{n} + 2\pi\kappa d \Sigma \{C_n I_1(nvd) - D_n K_1(nvd)\}.$$

2. Prove that the amount of heat per second that crosses unit area of the inner surface averages cha_0/d.

8·5. Transverse vibrations of a tapered rod.

In certain branches of applied mathematics the square of Laplace's operator makes its appearance; this is notably the case in elasticity and in the motion of a viscous liquid. The fundamental equation is then of the fourth order and usually requires more than two different types of Bessel function for its satisfaction. We begin with a relatively simple case.

Problem 16.—Discuss the transverse vibrations of a tapered rod.

In the discussion of the transverse vibrations of a taut string, we assume the absence of flexural rigidity and we ascribe the vibrations to the tension. The essential difference in the parallel discussion of a rod is that the flexural rigidity is taken to be paramount, whilst the longitudinal stresses are more or less ignored.

Imagine a thin rod to occupy part of the x axis and let AB be δx, an element of length, the origin for the moment remaining unspecified. Let a be the cross-section at A; I the moment of inertia of the section; E Young's modulus for the material; ρ the density, M the bending moment and Q the shearing force. In the approximate theory of bending, the shearing force is the negative gradient of the bending moment. The latter is given by Euler's formula and we have

$$-Q = \frac{\partial M}{\partial x}; \quad M = EI\frac{\partial^2 y}{\partial x^2},$$

where y is the deflection. The mass of the element AB is $\rho a\,\delta x$ and its acceleration $\partial^2 y/\partial t^2$ is caused by the element of shearing force δQ. This gives

$$\rho a\frac{\partial^2 y}{\partial t^2}\,\delta x = -\frac{\partial^2 M}{\partial x^2}\,\delta x = -\frac{\partial^2}{\partial x^2}\left\{EI\frac{\partial^2 y}{\partial x^2}\right\}\delta x.$$

Since oscillatory motion is possible we put $y = X\exp i\omega t$, where X is independent of t. This gives

$$\frac{d^2}{dx^2}\left\{EI\frac{d^2 X}{dx^2}\right\} = \rho a\omega^2 X.$$

In the simplest case where all but x and X are constants, the equation is soluble in terms of trigonometrical and hyperbolic functions, equivalent to Bessel functions of order $\frac{1}{2}$, and the matter is treated in various texts on sound, structures, elasticity, or strength of materials.

The most likely departure from uniformity will come from the cross-section and its modulus. There is then no standard technique for solving the equation; each particular case calls for individual treatment, with the result that considerable ingenuity is often required for effecting a solution. If, as an illustration, we assume the rod to be uniformly tapered, we can put $a = Ax^2$ and $I = Hx^4$. The equation then becomes

$$\frac{d^2}{dx^2}\left\{x^4 \frac{d^2X}{dx^2}\right\} = k^4x^2X, \quad k^4 = \frac{A\rho\omega^2}{HE}.$$

At this point, two methods of procedure present themselves; but before advancing, it is possibly as well to recall certain features in the solution of linear differential equations.

The equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

can be written indifferently as

$$(D-2)(D-3)y = 0, \quad (D-3)(D-2)y = 0, \quad D = \frac{d}{dx}.$$

The operators $(D-2)$ and $(D-3)$ happen to be permutable. This is by no means always the case, as the reader can readily verify from the two operators xD and x^2D. As a result of their permutability, the original equation is necessarily satisfied by any solution of either of the equations

$$(D-3)y = 0, \quad (D-2)y = 0,$$

i.e. by Ae^{3x} and Be^{2x}. As only two independent solutions are required for a second order equation the given equation is solved.

Alternatively, certain equations possess a type of homogeneity that may be illustrated by

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 4y = 0.$$

The substitution

$$x = e^\theta, \quad x\frac{d}{dx} = \frac{d}{d\theta} = \vartheta$$

leads to an operator ϑ whose properties may be summarized as

$$x^3\frac{d^3}{dx^3} = \vartheta(\vartheta-1)(\vartheta-2), \quad \vartheta(e^{a\theta}H) = e^{a\theta}(\vartheta + a)H.$$

The above equation would then be written

$$\{\vartheta(\vartheta-1)+3\vartheta+4\}y=0$$

and solved as a linear equation with constant coefficients.

Applying the former method in the present instance, it is readily verified that our equation can be written

$$\{x^2D^4+8xD^3+12D^2-k^4\}X=0.$$

The only hope of expressing this in two permutable operators is to write

$$(xD^2+aD+k^2)(xD^2+aD-k^2)X=0.$$

It then appears that the two forms are consistent if $a=3$ and we have

$$(xD^2+3D+k^2)(xD^2+3D-k^2)X=0.$$

It follows that a permissible form of X is a solution of either of the equations

$$\frac{d^2X}{dx^2}+\frac{3}{x}\frac{dX}{dx}+\frac{k^2}{x}X=0=\frac{d^2X}{dx^2}+\frac{3}{x}\frac{dX}{dx}-\frac{k^2}{x}X.$$

We accordingly have

$$X=x^{-1}\{AJ_2(2kx^{\frac{1}{2}})+BY_2(2kx^{\frac{1}{2}})+CI_2(2kx^{\frac{1}{2}})+DK_2(2kx^{\frac{1}{2}})\}.$$

Using the second method, which only differs symbolically from the former, we should write

$$e^{-2\theta}\vartheta(\vartheta-1)\{e^{2\theta}\vartheta(\vartheta-1)\}X=k^4e^{2\theta}X,$$

whence

$$e^{-2\theta}(\vartheta+2)(\vartheta+1)\vartheta(\vartheta-1)X=k^4X.$$

By a slight rearrangement we derive

$$e^{-\theta}\vartheta(\vartheta+2)e^{-\theta}\vartheta(\vartheta+2)X=k^4X.$$

It follows that any solution of either of the two equations

$$\{e^{-\theta}\vartheta(\vartheta+2)+k^2\}X=0=\{e^{-\theta}\vartheta(\vartheta+2)-k^2\}X$$

is satisfactory, and it can readily be verified that these are nothing but a disguised form of the previous pair, leading to the same result.

By differentiation we then have

$$\frac{dX}{dx} = -kx^{-3/2}\{AJ_3 + BY_3 - CI_3 + DK_3\},$$

$$\frac{d^2X}{dx^2} = +k^2x^{-2}\{AJ_4 + BY_4 + CI_4 + DK_4\},$$

the unwritten argument being in every case $2kx^{\frac{1}{2}}$. The final determination of k, and thence ω, is decided by whatever boundary conditions we adopt. If, for the sake of illustration, one end is defined by $x = 0$ and is free, we discard Y_4 and K_4 as giving infinity at the origin. If the other end is clamped and defined by $x = \lambda$, so that $X = 0 = X'$, we have

$$AJ_2(2k\lambda^{\frac{1}{2}}) + CI_2(2k\lambda^{\frac{1}{2}}) = 0,$$
$$AJ_3(2k\lambda^{\frac{1}{2}}) - CI_3(2k\lambda^{\frac{1}{2}}) = 0,$$

whence

$$\frac{J_2(2k\lambda^{\frac{1}{2}})}{I_2(2k\lambda^{\frac{1}{2}})} + \frac{J_3(2k\lambda^{\frac{1}{2}})}{I_3(2k\lambda^{\frac{1}{2}})} = 0$$

as the transcendental equation for the determination of the normal modes of vibration.

EXERCISES

1. In the last equation, show that J_2 and J_3 must have opposite signs and deduce that the equation has an infinite number of roots.

2. If the rod is of circular section, the radius being proportional to $x^{4/5}$, prove that the solution involves functions of order 5/6 with arguments proportional to $x^{3/5}$.

3. Investigate the general case where a is proportional to x^n and I is proportional to x^m. Deduce that the problem is soluble when

$$3m = n + 8 \quad \text{or} \quad m = n + 2.$$

This does not exhaust the possibilities.

8·6. Buckling of a circular disc.

The mathematical theory of elasticity has to be classed as a definitely difficult subject, and it is not rendered easier by the Cartesian method of approach, which happens to be the way it developed. The analysis required for the discussion of further problems is too lengthy to transcribe here. The reader must accordingly be asked to accept the adopted equations on the authority of the textbooks where he can find their justification and master it at leisure, unless he happens to be in the fortunate position of being already familiar with it. With this reservation we can continue with our applications.

Problem 17.—A uniform circular disc, whose middle plane is the z plane, has thickness 2h and radius a. The periphery is subjected to uniform radial pressure P per unit area. It is required to investigate the critical value of P which causes the disc to buckle.

The problem is in some respects the two-dimensional analogue of the strut but the analysis is more complicated. If we suppose that a point whose co-ordinates were r, ϕ, 0 moves to r, ϕ, z, we can discard ϕ in virtue of the symmetry about the z axis. It is then shown in the textbooks that the equilibrium equation has the form

$$\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\right\}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)+k^2\right\}z=0.$$

Adopting the substitution $r = e^\theta$, and writing ϑ for $d/d\theta$ as before, the equation takes the very simple form

$$(e^{-2\theta}\vartheta^2)(e^{-2\theta}\vartheta^2+k^2)z=0.$$

The operators are evidently permutable, so that any solution of either of the equations

$$e^{-2\theta}\vartheta^2 z=0, \quad (e^{-2\theta}\vartheta^2+k^2)z=0$$

will be acceptable. The former gives

$$z = A\theta + B = A\log r + B.$$

The latter equation is nothing but Bessel's equation of order zero,

$$(e^{-2\theta}\vartheta^2+k^2)z=\frac{d^2z}{dr^2}+\frac{1}{r}\frac{dz}{dr}+k^2z=0.$$

We are thus in possession of the full solution

$$z = A\log r + B + CJ_0(kr) + DY_0(kr)$$

containing the requisite number of arbitrary constants. Hence also

$$\frac{dz}{dr}=Ar^{-1}-kCJ_1(kr)-kDY_1(kr).$$

If the disc is complete, without a central hole, the values of z and dz/dr

must not be infinite at the centre. Since Y_1 goes to infinity logarithmically at the origin, we must have both A and D zero, the solution reducing to

$$z = B + CJ_0(kr), \quad \frac{dz}{dr} = -kCJ_1(kr).$$

The arbitrary constants are determined from the boundary conditions, there being two standard cases according as the rim is clamped or not. We propose to deal with the latter case.

When the boundary is not clamped there is neither displacement nor bending moment at the periphery. Following the textbooks, these are covered by

$$z = 0 = \frac{d^2z}{dr^2} + \frac{\sigma}{r}\frac{dz}{dr}, \quad r = a,$$

where σ denotes Poisson's ratio. The former condition is equivalent to

$$B + CJ_0(ka) = 0.$$

As for the latter condition, the equation for the Bessel function shows that it is equivalent to

$$kaJ_0(ka) = (1 - \sigma)J_1(ka).$$

As Poisson's ratio is usually about 0·3 and not a subject for meticulous accuracy, the equation is quite easy to solve with the aid of the tables. It has an infinity of roots; but all except the smallest would correspond to highly unstable configurations. The formula for z shows that the cross-section of the middle plane looks like the first half-loop of J_0 and its reflection. The constant B plays no part and might have been dropped; it corresponds to a bodily displacement of the disc and has no stress effect. The constant C remains indeterminate and gives a scale effect. It is of no consequence since if the buckling load is reached the damage is usually done. There remains the constant k, a convenient abbreviation defined by

$$k^2 = \frac{3P(1 - \sigma^2)}{Eh^2}.$$

EXERCISES

1. Taking Poisson's ratio as 0·3, solve the critical equation

$$xJ_0(x) = (1 - \sigma)J_1(x)$$

by means of the extracted values

x	J_0	J_1
2·04	0·2009	0·5738
2·05	0·1951	0·5730.

Hence compute for a circular disc of diameter 10 cm. and thickness 1 mm. taking Young's modulus as 2 10^9 gr./cm^2.

2. In the case of a clamped disc the boundary condition is evidently $\frac{dz}{dr} = 0.$ Prove that this leads to $J_1(ka) = 0$ whose smallest root is 3·8317. Hence compute for the disc as above.

8·7. Vibrations of a disc.

Since we have been considering a disc, the natural transition is to the discussion of its normal modes of vibration. A disc differs from a membrane as a rod differs from a string; the flexural rigidity plays the predominant part. In polar co-ordinates r, ϕ the vibrations of a disc are governed by the equation

$$\rho \frac{\partial^2 z}{\partial t^2} + \frac{Eh^2}{3(1-\sigma^2)} \left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right\}^2 z = 0.$$

The significance of the symbols is the same as before, with the addition of ρ which is the density. The analysis leading up to the equation can be found in Rayleigh, *Theory of Sound*, Vol. 1, § 218 and elsewhere. The problem is in some respects the two-dimensional analogue of the transversely vibrating rod. The boundary conditions for a clamped edge are simple. Since there is neither displacement nor slope we have

$$z = 0 = \frac{\partial z}{\partial r}.$$

In the case of a free edge, which implies the absence of shear and bending, the conditions are more complicated. They are usually given as

$$\frac{\partial^2 z}{\partial r^2} + \sigma\left\{\frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \phi^2}\right\} = 0,$$

$$\frac{\partial}{\partial r}\left\{\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \phi^2}\right\} + \frac{1-\sigma}{r^2}\frac{\partial^2}{\partial \phi^2}\left\{\frac{\partial z}{\partial r} - \frac{z}{r}\right\} = 0.$$

We assume, as usual, that the normal modes of vibration can be accounted for by the adoption of a periodic time factor $\exp i\omega t$. For the azimuthal angle ϕ we assume the presence of a factor $\exp in\phi$ with a view to Fourier analysis. We accordingly make the substitution

$$z = R\exp i(\omega t + n\phi),$$

where R is a function of r alone. The equation can then be written

$$\left\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right\}^2 R = k^4 R, \quad k^4 = \frac{3\rho\omega^2(1-\sigma^2)}{Eh^2}.$$

Alternatively

$$\{e^{-2\theta}(\vartheta^2 - n^2)\}^2 R = k^4 R,$$

which is clearly equivalent to the pair of equations

$$\{e^{-2\theta}(\vartheta^2 - n^2)\}R = \pm k^2 R,$$

and any solution of either of these is acceptable. Reverting to the independent variable r we have

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k^2 - \frac{n^2}{r^2}\right)R = 0 = \frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(k^2 + \frac{n^2}{r^2}\right)R.$$

The value of R is thus given by

$$R = AJ_n(kr) + BY_n(kr) + CI_n(kr) + DK_n(kr).$$

The fulfilment of the boundary conditions for any stipulated mode of support leads to a transcendental equation which ultimately determines ω and the frequency of the normal mode of vibration. With so many terms present, this may occasionally call for a certain amount of manipulative skill. As an illustration we consider a disc which is complete and free, meaning at rest and unsupported in a gravitationless field.

The disc being complete, the displacement of the centre must be finite and the slope there zero, as already explained. This compels us to discard the functions Y_n and K_n, and to work with the more limited solution

$$R = AJ_n(kr) + CI_n(kr),$$

equivalent to

$$z = \{AJ_n(kr) + CI_n(kr)\} \sin n\phi \cos \omega t.$$

The first of the two boundary conditions can be rewritten as

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{n^2}{r^2}R - \frac{(1-\sigma)}{r^2}\left\{r\frac{dR}{dr} - n^2R\right\} = 0.$$

As far as $J_n(kr)$ is concerned, and in virtue of the equation which it satisfies, this provides us with

$$-k^2J_n(kr) - \frac{(1-\sigma)}{r^2}\{krJ_n'(kr) - n^2J_n(kr)\}.$$

Similarly $I_n(kr)$ provides us with

$$k^2I_n(kr) - \frac{(1-\sigma)}{r^2}\{krI_n'(kr) - n^2I_n(kr)\}.$$

These can be modified by using the recurrence relations

$$xJ_n' = nJ_n - xJ_{n+1}; \quad xI_n' = nI_n + xI_{n+1}.$$

We conclude that the first boundary condition at $r = a$ is equivalent to

$$-\frac{C}{A} = \frac{(n^2 - n - m)J_n(ka) + kaJ_{n+1}(ka)}{(n^2 - n + m)I_n(ka) - kaI_{n+1}(ka)}, \quad m = \frac{k^2a^2}{1-\sigma}.$$

The secondary boundary condition can be written

$$\frac{d}{dr}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{n^2}{r^2}R\right] - \frac{n^2(1-\sigma)}{r^2}\left\{\frac{dR}{dr} - \frac{R}{r}\right\} = 0.$$

The contribution from $J_n(kr)$ is

$$-k^2\frac{dJ_n}{dr} - \frac{n^2(1-\sigma)}{r^2}\left\{\frac{dJ_n}{dr} - \frac{J_n}{r}\right\}.$$

By the use of the recurrence formula we have, after a little reduction,

$$\frac{1-\sigma}{r^3}\{n(n - m - n^2)J_n(kr) + (m + n^2)krJ_{n+1}(kr)\}.$$

Similarly we can show that the contribution from $I_n(kr)$ is

$$\frac{1-\sigma}{r^3}\{n(n+m-n^2)I_n(kr)+(m-n^2)krI_{n+1}(kr)\}.$$

The elimination of the ratio $-C/A$ then gives

$$\frac{(n^2-n-m)J_n(c)+cJ_{n+1}(c)}{(n^2-n+m)I_n(c)-cI_{n+1}(c)}=\frac{n(n^2-n+m)J_n(c)-(n^2+m)cJ_{n+1}(c)}{n(n^2-n-m)I_n(c)+(n^2-m)cI_{n+1}(c)},$$

where the argument is $c = ka$. This determines k and thence ω and the normal modes of vibration for a given value of n.

EXERCISES

1. Verify that the last relation can be written

$$c+\frac{2n^2(n-1)}{c}\frac{I_nJ_n}{I_{n+1}J_{n+1}}$$
$$=\frac{n^4-n^2+m^2}{2m}\left\{\frac{I_n}{I_{n+1}}+\frac{J_n}{J_{n+1}}\right\}+n(n-1)\left\{\frac{I_n}{I_{n+1}}-\frac{J_n}{J_{n+1}}\right\}.$$

2. Deduce that when there are no nodal diameters, the relation becomes

$$ka\left\{\frac{I_0(ka)}{I_1(ka)}+\frac{J_0(ka)}{J_1(ka)}\right\}=2(1-\sigma).$$

Establish this result independently.

3. In connexion with the last equation, prove that the function $xI_0(x)/I_1(x)$ has the value 2 when x is zero and steadily increases with x. Hence prove that the equation can have no root smaller than the first zero of $J_0(x)$, but must have a root before the first zero of $J_1(x)$. With $\sigma = 0{\cdot}3$ solve by the aid of the extracted values

x	J_0	J_1	I_0	I_1
2·99	−0·25664	0·34278	4·8414	3·9179
3·00	−0·26005	0·33906	4·8808	3·9534.

Hence compute for a disc of radius 2 cm. and thickness $\frac{1}{2}$ mm., density 7·6 gr. per c.c. and $E = 2\ 10^9$ gr. per cm.^{2}.

4. When there is a single nodal diameter, derive the relation from No. 1 above and establish it independently.

How much of the modified argument in No. 3 can now be applied as an aid to solution?

5. If the disc be complete and clamped at the rim, prove that the discriminating equation is

$$\frac{J_{n-1}(ka)}{J_n(ka)}=\frac{I_{n-1}(ka)}{I_n(ka)}.$$

Give an alternative form with orders n and $n+1$.

8·8. The non-uniform disc.

Having examined the vibrations of a uniform disc it is natural to examine the problem when the hypothesis of uniform thickness is abandoned for a certain variability. The law of variation needs to be fairly simple, say according to some power of the radius, and if we put $h = c(a/r)^s$ we can vary the rate of change of thickness, as we move along the radius, within wide limits. If s is negative the disc has zero thickness at the centre; but of course the disc need not be complete. Similarly if s is positive the central thickness is infinite. The disc may then either be incomplete or be considered as mounted at the end of a spindle.

The curious fact emerges that the solubility of the problem by Bessel functions depends on the number of nodal lines, and one particular case comes out in terms of the elementary functions. The fundamental equation is given in the texts as

$$\frac{E}{1-\sigma^2}\left\{\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right\}I\left\{\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 z}{\partial\phi^2}\right\}+2\rho h\frac{\partial^2 z}{\partial t^2}=0,$$

which closely resembles the corresponding equation for a non-uniform rod. In the present case we have

$$h = c\left(\frac{a}{r}\right)^s, \quad I = \tfrac{2}{3}h^3,$$

so that we can rewrite as

$$\left\{\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}\right\}r^{-3s}\left\{\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\frac{m^2R}{r^2}\right\}=k^4Rr^{-s},$$

where

$$z = R\sin m\phi\cos\omega t, \quad k^4=\frac{3\rho\omega^2(1-\sigma^2)}{Ec^2a^{2s}}.$$

Further progress is greatly facilitated by the symbolical method, so that

$$e^{-2\theta}(\vartheta^2-m^2)e^{-\theta(3s+2)}(\vartheta^2-m^2)R=k^4Re^{-s\theta},$$

whence

$$e^{-2\theta(s+2)}\{(\vartheta-3s-2)^2-m^2\}(\vartheta^2-m^2)R=k^4R.$$

The hope of a solution lies in expressing the operator as a square; but unless there is some re-arrangement of the factors we shall merely be left with the case where s is zero, as the reader can easily verify. We accordingly re-arrange the factors in the order

$$\vartheta - m - 3s - 2, \quad \vartheta - m, \quad \vartheta + m - 3s - 2, \quad \vartheta + m.$$

We can then write the equation as

$$e^{-\theta(s+2)}(\vartheta - m - 2s)(\vartheta - m + s + 2)e^{-\theta(s+2)}(\vartheta + m - 3s - 2)(\vartheta + m)R = k^4R.$$

The operator is now a square provided $2m = s + 2$. We accordingly remove s and write

$$\{e^{-2m\theta}(\vartheta - 5m + 4)(\vartheta + m)\}^2 R = k^4 R,$$

which is equivalent to the two equations

$$\frac{d^2R}{dr^2} + \frac{5 - 4m}{r}\frac{dR}{dr} + \left\{\frac{m(4 - 5m)}{r^2} \pm k^2r^{2m-2}\right\}R = 0.$$

These are plainly comparable with our standards 3·6(2) and 7·3(1). The comparison gives

$$1 - 2\alpha = 5 - 4m, \qquad \gamma = m,$$
$$\beta\gamma = k, \qquad \alpha^2 - n^2\gamma^2 = m(4 - 5m),$$

whence we derive

$$\alpha = 2m - 2 = s, \quad \beta = \frac{k}{m}, \quad n = \frac{3m - 2}{m}.$$

It appears that this gives no solution when m is zero, corresponding to a thickness proportional to the square of the radius. The next permissible value of m is unity and leads to

$$m = 1 = n = \gamma, \quad \alpha = 0 = s, \quad \beta = k.$$

The disc is of uniform thickness and the solution is given in functions of the type $J_1(kr)$. A certain interest attaches to the value $m = 4$. It leads to $\gamma = 4$, $\alpha = 6 = s$, $\beta = k/4$, $n = 5/2$, so that the functions are of order half an odd integer.

EXERCISES

1. Prove that the functions are of integral order only when m is 1 or 2. Prove also that the functions are of order half an odd integer only when m is 4.

2. By writing the factors in the order

$$\vartheta + m - 3s - 2, \quad \vartheta + m, \quad \vartheta - m - 3s - 2, \quad \vartheta - m,$$

prove that the operator can be made a square provided $2 + 2m + s = 0$. This leads to

$$\frac{d^2R}{dr^2} + \frac{4m+5}{r}\frac{dR}{dr} - \left\{\frac{m(5m+4)}{r^2} \pm \frac{k^2}{r^{2m+2}}\right\}R = 0.$$

Discuss the various types of function involved in the solution under different conditions, proving incidentally that m cannot be zero and that functions of order $3\frac{1}{3}$ may appear; also that only two values of m can lead to functions of integral order. Note that the argument of the functions decreases with increasing r.

3. Prove that the only other re-arrangement of the factors that will lead to a square operator is to put $(\vartheta^2 - m^2)$ first. Show that this gives s the unique value -1 and the thickness is proportional to the radial distance. Verify that the equations are

$$\frac{d^2R}{dr^2} + \frac{3}{r}\frac{dR}{dr} + \frac{1-m^2}{r^2} \pm \frac{k^2}{r} = 0.$$

These are soluble for all values of m and lead to functions of the type

$$r^{-1}J_{2m}(2kr^{\frac{1}{2}}).$$

CHAPTER IX

Bessel Coefficients. Integrals and Expansions

9·1. Bessel coefficients.

The functions $J_n(x)$ of integral order are sometimes called Bessel coefficients since they occur as coefficients in a certain expansion which we shall now consider. The left side of the identity

$$\exp\{\tfrac{1}{2}x(t - t^{-1})\} \equiv \exp(\tfrac{1}{2}xt)\exp(-\tfrac{1}{2}xt^{-1})$$

can be expressed as a both ways infinite series of ascending and descending powers of t, a so-called Laurent series. We take the product of the two expansions

$$\exp(\tfrac{1}{2}xt) = 1 + (\tfrac{1}{2}x)\frac{t}{1!} + (\tfrac{1}{2}x)^2\frac{t^2}{2!} + \ldots + (\tfrac{1}{2}x)^n\frac{t^n}{n!} + \ldots,$$

$$\exp(-\tfrac{1}{2}xt^{-1}) = 1 - (\tfrac{1}{2}x)\frac{t^{-1}}{1!} + (\tfrac{1}{2}x)^2\frac{t^{-2}}{2!} + \ldots (-\tfrac{1}{2}x)^n\frac{t^{-n}}{n!} + \ldots.$$

The independent term in this product is

$$1 - \frac{(\frac{1}{2}x)^2}{(1!)^2} + \frac{(\frac{1}{2}x)^4}{(2!)^2} - \ldots = J_0(x).$$

The coefficient of t^n is

$$\frac{(\frac{1}{2}x)^n}{n!}\left\{1 - \frac{(\frac{1}{2}x)^2}{1(n+1)} + \frac{(\frac{1}{2}x)^4}{1.2(n+1)(n+2)} - \ldots\right\} = J_n(x).$$

Similarly the coefficient of t^{-n} is

$$\frac{(-\frac{1}{2}x)^n}{n!}\left\{1 - \frac{(\frac{1}{2}x)^2}{1(n+1)} + \frac{(\frac{1}{2}x)^4}{1.2(n+1)(n+2)} - \ldots\right\} = (-)^n J_n(x) = J_{-n}(x).$$

We accordingly have

$$\exp\{\tfrac{1}{2}x(t - t^{-1})\} = \Sigma_{-\infty}^{\infty} t^n J_n(x)$$

$$= J_0(x) + J_1(x)(t - t^{-1}) + J_2(x)(t^2 + t^{-2}) + \ldots.$$

This last result is extremely fruitful and we proceed to make deductions from it. Putting $t = \exp i\theta$ we have

$$\exp(ix \sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \ldots$$
$$+ 2i\{J_1(x)\sin\theta + J_3(x)\sin 3\theta + \ldots\}.$$

Separation of the real and imaginary parts in

$$\exp(ix\sin\theta) = \cos(x\sin\theta) + i\sin(x\sin\theta)$$

gives the two series

(1) $\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \ldots,$

(2) $\sin(x\sin\theta) = 2\{J_1(x)\sin\theta + J_3(x)\sin 3\theta + \ldots\}.$

Replacing θ by its complement $\frac{1}{2}\pi - \theta$ we have correspondingly

(3) $\cos(x\cos\theta) = J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta - \ldots,$

(4) $\sin(x\cos\theta) = 2\{J_1(x)\cos\theta - J_3(x)\cos 3\theta + \ldots\}.$

These series are usually associated with the name of Jacobi. Any one of the four may be regarded as a Fourier series. Two are in even cosines; the others are in odd sines and odd cosines respectively. The law for the formation of the coefficients then gives

(5) $$\pi J_{2n}(x) = \int_0^\pi \cos(x\sin\theta)\cos 2n\theta\, d\theta = (-)^n\int_0^\pi \cos(x\cos\theta)\cos 2n\theta\, d\theta,$$

(6) $$\pi J_{2n+1}(x) = \int_0^\pi \sin(x\sin\theta)\sin(2n+1)\theta\, d\theta = (-)^n\int_0^\pi \sin(x\cos\theta)\cos(2n+1)\theta\, d\theta.$$

In each case the limits can be taken as 0 to $\frac{1}{2}\pi$ if we double the result. If θ be replaced by its supplement $\pi - \theta$ in $\cos(x\sin\theta)\cos n\theta$ there is a change of sign if n is odd. Hence we have

(7) $$\int_0^\pi \cos(x\sin\theta)\cos n\theta\, d\theta = \pi J_n(x) \quad (n \text{ even}), \qquad = 0 \quad (n \text{ odd}),$$

similarly

$$\int_0^\pi \sin(x\sin\theta)\sin n\theta\, d\theta = 0 \quad (n \text{ even}), \qquad = \pi J_n(x) \quad (n \text{ odd}).$$

It follows by addition that

$$(8) \qquad \int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta = \pi J_n(x) \quad (n \text{ integral}).$$

These integrals are associated with the name of Bessel. The last of them is of historic interest as being the starting point of Bessel's investigations. It arose in connexion with an astronomical problem concerning what is known as the "eccentric anomaly". As an example of the application we have the following.

9·2. Application to the cycloid.

Problem 18.—*A circle of unit radius rolls on a straight line. The original point of contact describes a cycloid. Express this in a Fourier series.*

With the origin at the initial point of contact, the parametric coordinates of the tracing-point are

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta.$$

The curve has a base of 2π and is symmetrical about its middle ordinate; the series is therefore cosines only, so that

$$y = a_0 + a_1\cos x + a_2\cos 2x + \ldots .$$

The limits for x are 0 to 2π and the same holds for θ. Hence

$$a_n = \frac{2}{\pi}\int_0^\pi y\cos nx\,dx = \frac{2}{\pi}\int_0^\pi \cos(n\theta - n\sin\theta)(1-\cos\theta)^2\,d\theta.$$

It remains to evaluate this integral, which we shall do by using the identity

$$(1-\cos\theta)^2 = 2 - 2\cos\theta - \sin^2\theta.$$

If we differentiate Bessel's last integral with respect to x we have

$$\int_0^\pi \sin(n\theta - x\sin\theta)\sin\theta\,d\theta = \pi J_n'(x).$$

A second differentiation gives

$$\int_0^\pi \cos(n\theta - x\sin\theta)\sin^2\theta\,d\theta = -\pi J_n''(x).$$

Moreover,

$$2\int_0^\pi \cos(n\theta - x\sin\theta)\cos\theta\, d\theta$$

$$= \int_0^\pi \cos\{(n+1)\theta - x\sin\theta\}d\theta + \int_0^\pi \cos\{(n-1)\theta - x\sin\theta\}d\theta$$

$$= \pi J_{n+1}(x) + \pi J_{n-1}(x)$$

$$= \frac{2\pi n}{x} J_n(x).$$

The Fourier coefficients are therefore given by

$$\tfrac{1}{2}\pi a_n = 2\pi J_n(n) - \frac{2\pi n}{n} J_n(n) + \pi J_n''(n)$$

or,

$$a_n = 2J_n''(n).$$

If we put $x = n$ in Bessel's equation we have

$$J_n''(n) + \frac{1}{n} J_n'(n) + \left(1 - \frac{n^2}{n^2}\right) J_n(n) = 0,$$

whence

$$a_n = -\frac{2}{n} J_n'(n).$$

The solution is completed by verifying that the mean ordinate a_0 is 3/2. The required Fourier series is therefore

$$y = \tfrac{3}{2} - 2\{J_1'(1)\cos x + \tfrac{1}{2}J_2'(2)\cos 2x + \tfrac{1}{3}J_3'(3)\cos 3x + \ldots\}.$$

EXERCISES

1. Obtain the recurrence formulæ by differentiating the relation

$$\exp\{\tfrac{1}{2}x(t - t^{-1})\} = \Sigma_{-\infty}^{\infty} t^n J_n(x)$$

with respect to x, or t, and comparing the coefficients of corresponding powers of t.

2. By modifying the variables in the above relation, prove that

$$\exp\{\tfrac{1}{2}x(t + t^{-1})\} = \Sigma_{-\infty}^{\infty} t^n I_n(x).$$

Replace t by t^{-1} and deduce $I_n(x) = I_{-n}(x)$.

3. From the Jacobi series deduce

$$1 = J_0(x) + 2J_2(x) + 2J_4(x) + \dots ,$$
$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \dots ,$$
$$\sin x = 2\{J_1(x) - J_3(x) + J_5(x) - \dots \}.$$

Establish corresponding results for the hyperbolic functions cosh and sinh. By means of one, two differentiations, prove that

$$\tfrac{1}{2}x = J_1(x) + 3J_3(x) + 5J_5(x) + \dots ,$$
$$\tfrac{1}{2}x \sin x = 2^2J_2(x) - 4^2J_4(x) + 6^2J_6(x) - \dots ,$$
$$\tfrac{1}{2}x \cos x = 1^2J_1(x) - 3^2J_3(x) + 5^2J_5(x) - \dots .$$

4. From the Fourier series for the cycloid, by taking the highest and lowest points, prove that

$$\tfrac{3}{4} = J_1'(1) + \tfrac{1}{2}J_2'(2) + \tfrac{1}{3}J_3'(3) + \dots ,$$
$$\tfrac{1}{4} = J_1'(1) - \tfrac{1}{2}J_2'(2) + \tfrac{1}{3}J_3'(3) - \dots .$$

Hence

$$\tfrac{1}{2} = J_1'(1) + \tfrac{1}{3}J_3'(3) + \tfrac{1}{5}J_5'(5) + \dots ,$$
$$\tfrac{1}{4} = \tfrac{1}{2}J_2'(2) + \tfrac{1}{4}J_4'(4) + \tfrac{1}{6}J_6'(6) + \dots .$$

5. A trochoid is described by a point whose distance from the centre is c, less than the radius which is unity. If the curve be expressed in a Fourier series, the lowest point being on the y axis, prove that the coefficient of $\cos nx$ is

$$a_n = -\frac{2c}{n} J_n'(nc).$$

The parametric co-ordinates are

$$x = \theta - c \sin \theta, \quad y = 1 - c \cos \theta.$$

[The problem on the cycloid introduces functions of equal order and argument. Considerable interest attaches to the theory of such functions, especially when the order is large. The convergence of the four series mentioned in Question 4 above is extraordinarily slow. Even when n is as high as 50, the value of $\frac{1}{n} J_n'(n)$ is about 0·0006.]

6. $J_0(x) = \dfrac{2}{\pi}\displaystyle\int_0^{\frac{1}{2}\pi} \cos(x \sin\theta)\, d\theta = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \exp(ix\cos\theta)\, d\theta,$

$$I_0(x) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cosh(x \sin\theta)\, d\theta = \frac{1}{\pi}\int_0^{\pi} \exp(x\cos\theta)\, d\theta.$$

7. Prove that the absolute value of $J_n(x)$ is less than unity.

8. Establish the relation

$$\exp(x\cos\theta) = I_0(x) + 2I_1(x)\cos\theta + 2I_2(x)\cos 2\theta + \dots .$$

9. If the anode voltage in a rectifying valve is $E \cos \omega t$ and the current may be taken as $I = A \exp(bE \cos \omega t)$, prove that the mean value of the current is $AI_0(bE)$ and that the root mean square is $A\{I_0(2bE)\}^{\frac{1}{2}}$.

9·3. The Poisson integral.

The function $J_n(x)$ can be defined by an integral, closely allied to the previous forms, which sometimes bears the name of Poisson and is otherwise referred to as Bessel's second integral. It is established as follows. The general, or $(r+1)$th, term in the expansion of $J_n(x)$ is

$$\frac{(-)^r(\frac{1}{2}x)^{n+2r}}{\Gamma(n+r+1)\Gamma(r+1)}, \quad r = 0, 1, 2, \ldots.$$

We propose now to multiply the expansion of $\cos(x\cos\theta)$ by $\sin^{2n}\theta$ and integrate termwise, n not necessarily being an integer. As groundwork for this we have from the Beta functions 1·6(2)

$$2\int_0^{\frac{1}{2}\pi} \cos^{2r}\theta \sin^{2n}\theta \, d\theta = \frac{\Gamma(r+\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(n+r+1)}.$$

Multiply top and bottom by $\Gamma(r+1)$ and use the duplication formula 1·8(1)

$$\Gamma(r+\tfrac{1}{2})\Gamma(r+1)2^{2r} = \sqrt{\pi}\Gamma(2r+1).$$

We then have, if r is an integer,

$$\int_0^{\pi} \cos^{2r}\theta \sin^{2n}\theta \, d\theta = 2\int_0^{\frac{1}{2}\pi} = \frac{\sqrt{\pi}\Gamma(n+\frac{1}{2})\Gamma(2r+1)}{2^{2r}\Gamma(n+r+1)\Gamma(r+1)}.$$

The $(r+1)$th term in the expansion of $\cos(x\cos\theta)$ is

$$\frac{(-)^r x^{2r}}{\Gamma(2r+1)} \cos^{2r}\theta, \quad r = 0, 1, 2, \ldots.$$

Multiply this by $\sin^{2n}\theta$ and integrate; we get

$$\frac{(-)^r x^{2r}}{\Gamma(2r+1)} \int_0^{\pi} \cos^{2r}\theta \sin^{2n}\theta \, d\theta = \frac{(-)^r(\frac{1}{2}x)^{2r}\Gamma(n+\frac{1}{2})\sqrt{\pi}}{\Gamma(n+r+1)\Gamma(r+1)}.$$

It will be observed that this is the corresponding term of $J_n(x)$ multiplied by

$$\Gamma(n+\tfrac{1}{2})\sqrt{\pi}(\tfrac{1}{2}x)^{-n}.$$

We conclude that

$$\text{(1)} \qquad J_n(x) = \frac{(\frac{1}{2}x)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}} \int_0^{\pi} \cos(x\cos\theta) \sin^{2n}\theta \, d\theta.$$

This is the formula referred to. The integrand is unchanged if θ be replaced by its supplement $\pi - \theta$, and we conclude that the integral can be doubled over the half-range, so that

$$J_n(x) \propto 2\int_0^{\frac{1}{2}\pi} \cos(x\cos\theta)\sin^{2n}\theta\, d\theta.$$

By using the complement $\frac{1}{2}\pi - \theta$ we have alternatively

$$J_n(x) \propto 2\int_0^{\frac{1}{2}\pi} \cos(x\sin\theta)\cos^{2n}\theta\, d\theta,$$

so that

$$(2) \qquad J_n(x) = \frac{(\frac{1}{2}x)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}}\int_0^{\pi} \cos(x\sin\theta)\cos^{2n}\theta\, d\theta.$$

9·4. Application to the circle.

Students of Fourier analysis, as performed by the integral calculus, are usually confined to diagrams consisting of straight lines, with occasional excursions into exponentials and parabolas. The circle never appears, and here is the explanation.

Problem 19.—*Express the upper half of the circle $x^2 + y^2 = \pi^2$ in a Fourier series.*

Taking the radius as π is a mere matter of convenience; by a change of scale it can later be made anything we like. The mean ordinate is $\frac{1}{4}\pi^2$ and since the curve is symmetrical about the y axis, the series is cosines only. We accordingly have

$$y = \tfrac{1}{4}\pi^2 + a_1\cos x + a_2\cos 2x + \ldots,$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} y\cos nx\, dx.$$

Turning to polar co-ordinates

$$x = \pi\cos\theta, \quad y = \pi\sin\theta,$$

we have

$$a_n = \frac{2}{\pi}\int_{\frac{1}{2}\pi}^{0} (\pi\sin\theta)\cos(n\pi\cos\theta)(-\pi\sin\theta\, d\theta)$$

$$= 2\pi\int_0^{\frac{1}{2}\pi} \cos(n\pi\cos\theta)\sin^2\theta\, d\theta.$$

Note the change of limits when θ replaces x. The integral is equivalent to

$$a_n = \pi \frac{\Gamma(1\frac{1}{2})\sqrt{\pi}}{(\frac{1}{2}n\pi)} J_1(n\pi) = \frac{\pi}{n} J_1(n\pi).$$

The required series is therefore

$$y = \frac{\pi^2}{4} + \pi\left\{\frac{J_1(\pi)}{1}\cos x + \frac{J_1(2\pi)}{2}\cos 2x + \frac{J_1(3\pi)}{3}\cos 3x + \ldots\right\}.$$

9·5. Modification of the integral.

A useful modification of the formulæ can be made by first transposing the factor x^n. We then differentiate with respect to x, taking advantage of the formula

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x).$$

Afterwards we replace the factor x^n. This gives

$$(1)\qquad J_{n+1}(x) = \frac{(\frac{1}{2}x)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}}\int_0^{\pi}\sin(x\cos\theta)\sin^{2n}\theta\cos\theta\,d\theta$$

$$(2)\qquad = \frac{(\frac{1}{2}x)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}}\int_0^{\pi}\sin(x\sin\theta)\cos^{2n}\theta\sin\theta\,d\theta.$$

9·6. The Lissajous figure of eight.

As an illustration of their use, consider the following variant of a familiar problem.

Problem 20.—A moving light-spot simultaneously executes the two perpendicular simple harmonic motions

$$x = \pi\cos\omega t, \quad y = c\sin 2\omega t.$$

It is required to find the Fourier series for its path in the time interval 0 to π/ω.

It is well known that the point repeatedly describes a figure of eight, one of the so-called Lissajous figures (fig. 9). The time interval π/ω is a vertical period, or half a horizontal period. Initially $t = 0 = y$, $x = \pi$. Finally $t = \pi/\omega$, $y = 0$, $x = -\pi$. At half-time we have $\omega t = \frac{1}{2}\pi$, $x = 0 = y$. The curve is skew and the requisite series is of the form

$$y = b_1\sin x + b_2\sin 2x + \ldots .$$

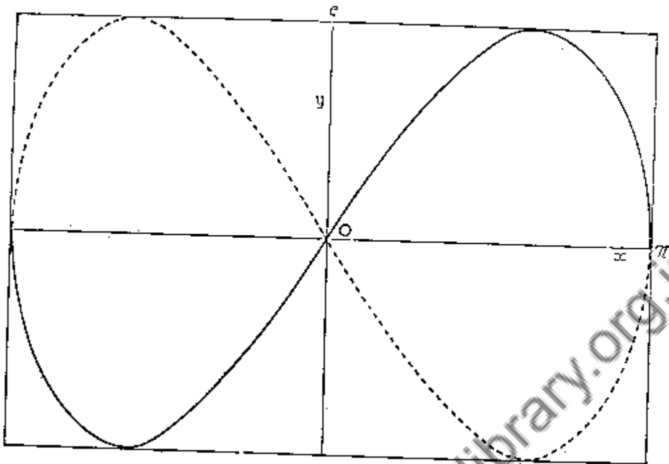

Fig. 9.—The Lissajous figure of eight

The coefficients are given by

$$b_n = \frac{2}{\pi}\int_0^\pi y \sin nx\,dx.$$

Put

$$\omega t = \theta, \quad x = \pi\cos\theta, \quad y = c\sin 2\theta, \quad dx = -\pi\sin\theta\,d\theta.$$

The limits for θ corresponding to $x = \pi$, 0 are $\theta = 0$, $\frac{1}{2}\pi$ respectively. Hence

$$b_n = \frac{2}{\pi}\int_{\frac{1}{2}\pi}^0 (c\sin 2\theta)\sin(n\pi\cos\theta)(-\pi\sin\theta\,d\theta)$$

$$= 2c\int_0^\pi \sin(n\pi\cos\theta)\sin^2\theta\cos\theta\,d\theta.$$

This is patently the upper of the two forms 9·5(1). Comparison then gives

$$b_n = 2c\left\{J_2(n\pi)\frac{\Gamma(1\frac{1}{2})\sqrt{\pi}}{(\frac{1}{2}n\pi)}\right\} = \frac{2c}{n}J_2(n\pi).$$

The required series is accordingly

$$\frac{y}{2c} = J_2(\pi)\sin x + \tfrac{1}{2}J_2(2\pi)\sin 2x + \tfrac{1}{3}J_2(3\pi)\sin 3x + \ldots .$$

The horizontal amplitude is changed from π to a if x be replaced by $\pi x/a$.

9·7. Transformation and an application to the circle.

Bessel's second integral admits of various transformations. They mostly amount to substituting, in the first form t for $\cos\theta$, in the second form t for $\sin\theta$. In either case we get

$$(1) \qquad J_n(x) = \frac{(\frac{1}{2}x)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}} \int_{-1}^{1} (1-t^2)^{n-\frac{1}{2}} \cos xt\, dt.$$

Here again the integral can be doubled over the half-range 0 to 1. Moreover, if we follow the same technique of first transposing the factor x^n and then differentiating with respect to x, we derive

$$(2) \qquad J_{n+1}(x) = \frac{(\frac{1}{2}x)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}} \int_{-1}^{1} t(1-t^2)^{n-\frac{1}{2}} \sin xt\, dt.$$

As an illustration of the application we offer the following.

Problem 21.—*It is required to find the Fourier series for the upper half of the circle defined in polar co-ordinates by the equation* $r = 2\pi \sin\theta$.

The diameter, as a matter of convenience, is 2π. The origin is on the circumference. The Cartesian co-ordinates are

$$x = 2\pi \sin^2\theta, \quad y = 2\pi \sin\theta \cos\theta.$$

The curve is symmetrical about its middle ordinate and the required series hence has the form

$$y = \tfrac{1}{4}\pi^2 + a_1 \cos x + a_2 \cos 2x + \cdots .$$

The coefficients are given by the rule

$$a_n = \frac{2}{\pi}\int_0^{\pi} y \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi/4} (\pi \sin 2\theta) \cos(2n\pi \sin^2\theta)(2\pi \sin 2\theta\, d\theta).$$

Further progress is facilitated by writing

$$\cos(2n\pi \sin^2\theta) = \cos(n\pi - n\pi\cos 2\theta)$$
$$= (-)^n \cos(n\pi \cos 2\theta).$$

The relation becomes

$$a_n = 4\pi \int_0^{\pi/4} (-)^n \cos(n\pi\cos 2\theta) \sin^2 2\theta\, d\theta.$$

We now make the substitution

$$\cos 2\theta = t, \quad \sin 2\theta = (1-t^2)^{\frac{1}{2}}, \quad -2\sin 2\theta\, d\theta = dt,$$

whence

$$a_n = -2\pi(-)^n \int_1^0 (1-t^2)^{\frac{1}{2}} \cos n\pi t\, dt.$$

Comparison then gives

$$a_n = \frac{(-)^n \Gamma(1\tfrac{1}{2})\sqrt{\pi}}{\tfrac{1}{2}n\pi} \cdot \pi J_1(n\pi) = \frac{(-)^n}{n} \pi J_1(n\pi).$$

This can be checked against the previous circle-problem by changing the origin.

EXERCISES

1. From the former of the two circle-problems establish the following identities:

$$\tfrac{1}{2} = J_1(\pi) + \tfrac{1}{3}J_1(3\pi) + \tfrac{1}{5}J_1(5\pi) + \ldots .$$
$$\tfrac{1}{4}\pi = J_1(\pi) - \tfrac{1}{2}J_1(2\pi) + \tfrac{1}{3}J_1(3\pi) - \ldots .$$
$$1 - \tfrac{1}{2}\pi = J_1(2\pi) + \tfrac{1}{2}J_1(4\pi) + \tfrac{1}{3}J_1(6\pi) + \ldots .$$

2. An elliptic arch has a span of 40 ft. and rise 18 ft. Taking centre line and ground level as axes, prove that the Fourier series, of period 40 ft. for its outline, is

$$\frac{y}{18} = \frac{\pi}{4} + J_1(\pi)\cos\frac{\pi x}{20} + \tfrac{1}{2}J_1(2\pi)\cos\frac{2\pi x}{20} + \ldots .$$

3. From the Lissajous figure of eight, deduce the identity

$$\frac{\sqrt{3}}{4} = J_2(\pi) - \tfrac{1}{3}J_2(3\pi) + \tfrac{1}{5}J_2(5\pi) - \ldots .$$

4. If the position of a moving point is defined at time t by the relations

$$x = a\cos\omega t, \quad y = c\sin 3\omega t,$$

prove that its path in the time interval $0 < \omega t < \frac{1}{2}\pi$ can be expressed as

$$-\frac{y}{3c} = J_3(\pi)\cos\frac{\pi x}{a} + \tfrac{1}{2}J_3(2\pi)\cos\frac{2\pi x}{a} + \tfrac{1}{3}J_3(3\pi)\cos\frac{3\pi x}{a} + \ldots .$$

5. From the last result above, deduce the identities:

$$\tfrac{1}{6} = J_3(\pi) + \tfrac{1}{3}J_3(3\pi) + \tfrac{1}{5}J_3(5\pi) + \ldots .$$
$$\tfrac{1}{6} = J_3(2\pi) + \tfrac{1}{2}J_3(6\pi) + \tfrac{1}{5}J_3(10\pi) + \ldots .$$
$$\tfrac{1}{3} = J_3(4\pi) + \tfrac{1}{2}J_3(8\pi) + \tfrac{1}{3}J_3(12\pi) + \ldots .$$

6. Multiply the identity $\cos 2\theta = 1 - 2\sin^2\theta$ by $\cos(x\cos\theta)$ and integrate from 0 to π, thus deducing $J_0 + J_2 = \dfrac{2}{x}J_1$. Treat similarly $\cos 4\theta = 1 - 8\sin^2\theta + 8\sin^4\theta$. Hence deduce generally that J_{2n} is expressible in terms of $J_0, J_1, \ldots J_n$.

7. Establish the relations

$$I_n(x) = \left\{\frac{2(\frac{1}{2}x)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}}\right\}\int_0^{\frac{1}{2}\pi}\cosh(x\sin\theta)\cos^{2n}\theta\,d\theta$$

$$= \{\quad\}\int_0^{\frac{1}{2}\pi}\cosh(x\cos\theta)\sin^{2n}\theta\,d\theta$$

$$= \{\quad\}\int_0^1 (1-t^2)^{n-\frac{1}{2}}\cosh tx\,dt.$$

9·8. Application of Lommel integrals.

We now transfer our attention to instances where Bessel functions figure in the integrand. The cases of Fresnel integrals and Lommel integrals were both mentioned earlier in the book; but as no application of the latter has yet been studied, it is desirable to fill the lacuna. This is best done by an example in heat conduction.

If ϕ be the temperature at any point, we have from 8·4(1)

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial\phi}{\partial r}\right\} + \frac{\partial^2\phi}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{\rho s}{\kappa}\frac{\partial\phi}{\partial t}.$$

If we can regard ϕ as independent of z and θ, the conduction takes place radially and we have

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{\rho s}{\kappa}\frac{\partial\phi}{\partial t}.$$

Assuming for a falling temperature that ϕ has the form $R\exp(-mt)$, where R is a function of r alone, we have solutions of the type

$$\phi = \{AJ_0(kr) + BY_0(kr)\}e^{-mt}, \quad k^2 = \frac{\rho s m}{\kappa}.$$

Problem 22.—A solid bar of radius c radiates into a medium at zero temperature. Assuming that the temperature in the bar is independent of the axial distance and of the azimuthal angle, it is required to examine the variation in temperature distribution with time.

Discarding the Y function because the bar is solid, we adopt solutions of the type

$$\phi = AJ_0(kr)e^{-mt}.$$

For the emissivity we have the surface condition

$$h\phi + \kappa \frac{\partial \phi}{\partial r} = 0, \quad r = c,$$

whence

$$hJ_0(kc) + \kappa k J_0'(kc) = 0,$$

or,

$$(1) \qquad J_0'(kc) + pJ_0(kc) = 0, \quad p = \frac{h}{\kappa k}.$$

The roots of this equation fix the permissible values of k, and thence m. If we build up a series solution

$$\phi = \Sigma A_1 J_0(k_1 r) \exp(-m_1 t),$$

the presumption is that initially the temperature distribution is given by

$$(2) \qquad \phi_0 = f(r) = \Sigma A_1 J_0(k_1 r).$$

The problem poses two questions at this stage. Firstly, whether an arbitrary function $f(r)$ can necessarily be expressed in a series of functions of the type $J_0(k_s r)$ where the k_s are assigned. We assume that it can, leaving the pure mathematicians to argue about the conditions of its validity. Secondly, how the coefficients A_s are to be determined; here we assume the legitimacy of termwise integration and we utilize Lommel's integrals. Consider the relation 4·6(1)

$$(\alpha^2 - \beta^2)\int_0^c rJ_0(\alpha r)J_0(\beta r)\,dr = c\{\beta J_0(\alpha c)J_0'(\beta c) - \alpha J_0'(\alpha c)J_0(\beta c)\}.$$

The right side is zero if α, β are two values of k determined from (1). It follows that we have $f(r)$ expressed in a series of orthogonal functions, and if we multiply (2) by $rJ_0(k_u r)$ and integrate, we have

$$A_v\int_0^c rJ_0(k_u r)J_0(k_v r)\,dr = 0, \quad u \neq v.$$

Turning to the companion relation 4·6(3)

$$\int_0^c rJ_0^2(kr)\,dr = \tfrac{1}{2}c^2\{J_0^2(kc) + J_0'^2(kc)\}$$

$$= \tfrac{1}{2}c^2(1 + p^2)J_0^2(kc),$$

we conclude that

$$\int_0^c rf(r)J_0(k_s r)\,dr = A_s \int_0^c rJ_0{}^2(k_s r)\,dr$$
$$= \tfrac{1}{2}A_s c^2(1 + p^2)J_0{}^2(k_s c).$$

The method of determining the coefficients is analogous to that employed in Fourier analysis. Once the equation for k_s has been solved, A_s may be considered as known. Whether one could perform the integration on the left in any particular case is another matter. Further illustrations will be found in H. S. Carslaw, *The Conduction of Heat*, Chap. VII.

9·9. Lipschitz's integral.

The number of known integrable expressions involving Bessel functions in the integrand is already somewhat extensive and there are accretions almost every year. Most of them require complex analysis for their establishment; but there are two exceptions which we will now give. The first is due to Lipschitz. As groundwork we require two elementary integrals.

(i) Treating $\int e^{-ax}\cos bx\,dx$ symbolically, we have

$$\frac{1}{D}\,e^{-ax}\cos bx = e^{-ax}\,\frac{1}{D-a}\cos bx$$

$$= -\frac{e^{-ax}}{a^2+b^2}\,(D+a)\cos bx = \frac{e^{-ax}}{a^2+b^2}\,(b\sin bx - a\cos bx).$$

It follows that for the infinite integral we have

$$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}.$$

(ii) If we write

$$\int \frac{dt}{a^2+b^2\cos^2 t} = \int \frac{d(\tan t)}{(a^2+b^2)+a^2\tan^2 t}$$
$$= \frac{1}{a(a^2+b^2)^{\frac{1}{2}}}\tan^{-1}\left\{\frac{a\tan t}{(a^2+b^2)^{\frac{1}{2}}}\right\},$$

we have the definite integral

$$\int_0^{\frac{1}{2}\pi} \frac{dt}{a^2+b^2\cos^2 t} = \frac{\pi}{2a(a^2+b^2)^{\frac{1}{2}}}.$$

Taking Bessel's first integral, coupled with a change in the order of integration, we have

$$\int_0^\infty e^{-ax}J_0(bx)\,dx = \int_0^\infty e^{-ax}\,dx\,\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\cos(bx\cos t)\,dt$$

$$= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}dt\int_0^\infty e^{-ax}\cos(bx\cos t)\,dx$$

$$= \frac{2a}{\pi}\int_0^{\frac{1}{2}\pi}\frac{dt}{a^2+b^2\cos^2 t}$$

$$= \frac{1}{(a^2+b^2)^{\frac{1}{2}}}.$$

EXERCISES

1. Prove

$$\int_0^\infty J_0(bx)\,dx = \frac{1}{b},\quad \int_0^\infty J_0(x)\,dx = 1.$$

2. Multiply the expansion of $J_0(bx)$ by e^{-ax} and integrate termwise, using the relation 1·8 Ex. 1,

$$\int_0^\infty e^{-ax}x^{n-1}\,dx = a^{-n}\Gamma(n).$$

Hence establish Lipschitz's integral.

3. Differentiate the integral with respect to b and deduce

$$\int_0^\infty e^{-ax}J_1(bx)x\,dx = \frac{b}{(a^2+b^2)^{3/2}}.$$

Transpose the factor b and differentiate again, utilizing the relation

$$\frac{d}{dt}\{t^{-n}J_n(t)\} = -t^{-n}J_{n+1}(t).$$

Establish that for n integral

$$\int_0^\infty e^{-ax}J_n(bx)x^n\,dx = \frac{(2b)^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(a^2+b^2)^{n+\frac{1}{2}}}.$$

$$\int_0^\infty e^{-ax}J_n(bx)x^{n+1}\,dx = \frac{2a(2b)^n\Gamma(n+3/2)}{\sqrt{\pi}(a^2+b^2)^{n+3/2}}.$$

Let a tend to zero. Is this legitimate?

9·10. Sonine's integral.

Sonine's first finite integral is readily established by the method previously employed for Bessel's first integral. We propose to multiply the expansion of $J_n(x \sin\theta)$ by $(\sin\theta)^{n+1}(\cos\theta)^{2m+1}$ and integrate termwise. The $(r+1)$th term in the expansion is

$$\frac{(-)^r(\frac{1}{2}x)^{n+2r}}{\Gamma(n+r+1)\Gamma(r+1)}(\sin\theta)^{n+2r},$$

so that the essential part of the integral is a Beta function and expressible in Gamma functions. We have

$$2\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2n+2r+1}(\cos\theta)^{2m+1}\,d\theta = \frac{\Gamma(n+r+1)\Gamma(m+1)}{\Gamma(n+r+m+2)}.$$

The reason for taking $(n+1)$ as the index of $\sin\theta$ now appears. It produces the factor $\Gamma(n+r+1)$ which cancels with the corresponding factor in the denominator. The net result of the integration is

$$\frac{(-)^r(\frac{1}{2}x)^{n+2r}\Gamma(m+1)}{\Gamma(r+1)\Gamma(n+r+m+2)}.$$

Inspection shows that if this were multiplied by

$$\frac{x^{m+1}}{2^m\Gamma(m+1)}$$

we should have the $(r+1)$th term of $J_{n+m+1}(x)$. We conclude that

$$(1)\quad J_{n+m+1}(x) = \frac{x^{m+1}}{2^m\Gamma(m+1)}\int_0^{\frac{1}{2}\pi}J_n(x\sin\theta)(\sin\theta)^{n+1}(\cos\theta)^{2m+1}\,d\theta.$$

EXERCISES

1. Prove that the validity of the result is not confined to integral values of m and n. Why is it necessary to stipulate that $m, n > -1$? Try the effect of putting $n = -\frac{1}{2}$ and replacing m by $m - \frac{1}{2}$.

2. Transform the integral by the substitution $\sin\theta = t$ and deduce

$$J_{n+m+1}(x) = \frac{x^{m+1}}{2^m\Gamma(m+1)}\int_0^1 J_n(xt)(1-t^2)^m t^{n+1}\,dt.$$

3. Deduce

$$\int_0^1 J_0(xt)t\,dt = x^{-1}J_1(x).$$

Hence by differentiation prove that

$$\int_0^1 J_1(xt)t^2\,dt = x^{-1}J_2(x).$$

Generalize by showing that Sonine's integral satisfies the relation

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x).$$

4. Evaluate the integral

$$\int_0^c J_0(kx)(c^2 - x^2)^n x\,dx.$$

5. Show that the equation

$$\int_0^1 \frac{\cos xt}{(1-t^2)^{\frac{1}{2}}}\,dt = 0$$

is satisfied when x is any zero of J_0.

9·11. Weber's discontinuous integrals.

If we give an imaginary value to a in Lipschitz's integral we have

$$\int_0^\infty e^{-iax}J_0(bx)\,dx = (b^2 - a^2)^{-\frac{1}{2}}.$$

The left side is complex; but the right side is either quite real or pure imaginary, according as b is greater or less than a. Equating the real and imaginary parts we have

$$\begin{aligned}\int_0^\infty J_0(bx)\cos ax\,dx &= 0, \quad a > b \\ &= (b^2 - a^2)^{-\frac{1}{2}}, \quad b > a. \\ \int_0^\infty J_0(bx)\sin ax\,dx &= (a^2 - b^2)^{-\frac{1}{2}}, \quad a > b \\ &= 0, \quad b > a.\end{aligned}$$

This is known as Weber's discontinuous integral; it is a fair sample of a whole hierarchy of discontinuous integrals.

Integrating the first form with respect to a from $a = p$ to $a = q$, we have

$$\int_0^\infty J_0(bx)\frac{\sin qx - \sin px}{x}\,dx = 0, \quad (p > b,\ q > b).$$

Hence the integral

$$\int_0^\infty J_0(bx)\frac{\sin ax}{x}\,dx (= I, \text{ say})$$

is independent of a, provided $a > b$. It remains to see whether I is a function of b. Substitute $ax = t$; then

$$I = \int_0^\infty J_0(ct)\,\frac{\sin t}{t}\,dt, \quad c = \frac{b}{a}.$$

This shows that I is a function of c only; but seeing that it is not a function of a, it is not a function of b either. We determine its value by letting b tend to zero, with $J_0(0) = 1$. We then have

$$I = \int_0^\infty \frac{\sin t}{t}\,dt = \tfrac{1}{2}\pi.$$

Incidentally there are at least fourteen independent proofs of this last relation. We conclude that (if $a > b$)

$$\int_0^\infty J_0(bx)\,\frac{\sin ax}{x}\,dx = \tfrac{1}{2}\pi.$$

Lipschitz's integral has found applications in the theory of potential. Sonine's integral and its near relations appear in some of Rayleigh's work, whilst Weber's integral was used in an investigation of the electrostatic potential of a charged disc. In the main, generalizations lead to the hypergeometric function, and possibly associated Legendre functions.

9·12. Fourier-Bessel series.

The type of series which we assumed in 9·8(2) as the expansion of an arbitrary function is known as a Fourier-Bessel expansion; it is a great stand-by in applied problems where a series expansion is required. All the functions are of the same order, not necessarily zero as in our case, and the arguments are determined from some equation of condition that permits the use of Lommel integrals. We take in general

$$f(x) = A_1 J_n(k_1 x) + A_2 J_n(k_2 x) + \cdots.$$

Multiply each side by $xJ_n(k_s x)$ and integrate from 0 to c. We have 4·6(1)

$$(\alpha^2 - \beta^2)\int_0^c xJ_n(\alpha x)J_n(\beta x)\,dx = c\{\beta J_n(\alpha c)J_n{}'(\beta c) - \alpha J_n{}'(\alpha c)J_n(\beta c)\}.$$

The right side is zero if α, β are values of k determined from

$$\text{(i)}\ J_n(kc) = 0,$$
$$\text{(ii)}\ J_n'(kc) = 0,$$
$$\text{(iii)}\ PkJ_n'(kc) + QJ_n(kc) = 0,$$

where P, Q are constants. The first of these was used by Fourier when n is zero. The extension to other orders was made by Lommel; and the third condition, which we employed in problem 9·8(1), is usually associated with the name of Dini.

The coefficients are then determined from the second of Lommel's integrals

$$\int_0^c xJ_n^2(\alpha x)\,dx = \tfrac{1}{2}c^2\left\{\left(1 - \frac{n^2}{\alpha^2c^2}\right)J_n^2(\alpha c) + J_n'^2(\alpha c)\right\},$$

which takes different forms according to the type of boundary condition employed. For example, if the first of the above conditions holds we have

$$\int_0^c xf(x)J_n(k_sx)\,dx = \tfrac{1}{2}A_sc^2J_n'^2(k_sc).$$

It has to be admitted that relatively few functions give compactly integrable forms. An obvious exception comes from

$$\int x^{n+1}J_n(kx)\,dx = \frac{x^{n+1}}{k}J_{n+1}(kx).$$

This shows that if $f(x)$ is x^n and we write

$$x^n = A_1J_n(k_1x) + A_2J_n(k_2x) + \ldots$$

we have

$$\frac{c^{n+1}}{k_s}J_{n+1}(k_sc) = \tfrac{1}{2}A_sc^2J_n'^2(k_sc), \qquad J_n(kc) = 0.$$

In virtue of the recurrence relation

$$kxJ_n'(kx) = nJ_n(kx) - kxJ_{n+1}(kx),$$

we have

$$A_s = \frac{2}{k_s}\frac{c^{n-1}}{J_{n+1}(k_sc)},$$

and the series for x^n is thus determined over a range 0 to c.

9·13. The polynomial.

It is sometimes an advantage to have all arguments the same and to allow the order of the function to vary. We accordingly consider the possibility of an expansion of the type

$$A_1J_1(x) + A_2J_2(x) + A_3J_3(x) + \dots .$$

In this connexion we can utilize the Jacobi series to give an expansion for x^n and hence for a polynomial. For brevity we put $2\cos\theta = c$. The trigonometry books then state that

$$2\cos 2\theta = c^2 - 2, \quad 2\cos 3\theta = c^3 - 3c,$$

and in general

$$2\cos n\theta = c^n - nc^{n-2} + \frac{n(n-3)}{2!}c^{n-4} - \frac{n(n-4)(n-5)}{3!}c^{n-6} + \dots .$$

We can write the Jacobi series as 9·1(1)

$$\cos\tfrac{1}{2}xc = J_0(x) - J_2(x)(c^2-2) + J_4(x)(c^4-4c^2+2) - \dots ,$$
$$\sin\tfrac{1}{2}xc = J_1(x)c - J_3(x)(c^3-3c) + J_5(x)(c^5-5c^3+5c) - \dots .$$

If in both cases we expand the left side in powers of xc, we can compare the coefficients of corresponding powers of c. In this way we derive, for the first three powers,

$$1 = J_0(x) + 2J_2(x) + 2J_4(x) + \dots ,$$
$$\tfrac{1}{2}x = J_1(x) + 3J_3(x) + 5J_5(x) + \dots ,$$
$$\tfrac{1}{8}x^2 = J_2(x) + 4J_4(x) + 9J_6(x) + \dots .$$

It can be left as an exercise for the reader to show that in general

$$(\tfrac{1}{2}x)^n = \Sigma_0^\infty \frac{(n+2r)(n+r-1)!}{r!} J_{n+2r}(x).$$

9·14. Schlömilch expansion.

Inspection of almost any vibration problem shows the occasional necessity for a series of uniform order, the argument being proportional to the rank of the term. This means that we seek an expansion of the form

$$f(x) = a_0 + a_1J_0(x) + a_2J_0(2x) + \dots .$$

If this were possible, we should have at the origin

$$f(0) - a_0 = \Sigma_1^\infty a_n.$$

We should also have by differentiation

$$-f'(x) = a_1 J_1(x) + 2a_2 J_1(2x) + 3a_3 J_1(3x) + \dots .$$

Presuming that we require this to be valid over the range $0 < x < c$, we substitute $x = c \sin\theta$; the limits for θ are then 0 to $\frac{1}{2}\pi$. We now propose to integrate termwise the expansion

$$J_1(nc\sin\theta) = (\tfrac{1}{2}nc)\sin\theta - (\tfrac{1}{2}nc)^3 \frac{\sin^3\theta}{1\,.\,2} + (\tfrac{1}{2}nc)^5 \frac{\sin^5\theta}{1\,.\,2\,.\,2\,.\,3} - \dots .$$

The general, or rth term is, apart from sign,

$$(\tfrac{1}{2}nc)^{2r-1} \frac{(\sin\theta)^{2r-1}}{\Gamma(r)\Gamma(r+1)}.$$

As part of the integration we have 1·6(2)

$$2\int_0^{\frac{1}{2}\pi} \sin^{2r-1}\theta\, d\theta = B(r, \tfrac{1}{2}) = \frac{\Gamma(r)\sqrt{\pi}}{\Gamma(r+\frac{1}{2})}.$$

The integral of our general term is thus

$$\frac{(\frac{1}{2}nc)^{2r-1}}{\Gamma(r)\Gamma(r+1)} \frac{1}{2} \frac{\Gamma(r)\sqrt{\pi}}{\Gamma(r+\frac{1}{2})}.$$

The duplication formula gives 1·8(1)

$$2^{2r}\Gamma(r+\tfrac{1}{2})\Gamma(r+1) = \sqrt{\pi}\,\Gamma(2r+1).$$

The integral of our general term becomes

$$\frac{(nc)^{2r-1}}{(2r)!}.$$

Hence

$$\int_0^{\frac{1}{2}\pi} J_1(nc\sin\theta)\, d\theta = \frac{nc}{2!} - \frac{(nc)^3}{4!} + \frac{(nc)^5}{6!} - \dots$$

$$= \frac{1 - \cos nc}{nc}.$$

Reverting to the series for $f'(x)$ we have

$$-\int_0^{\frac{1}{2}\pi} f'(c\sin\theta)\,d\theta = a_1\,\frac{1-\cos c}{c} + a_2\,\frac{1-\cos 2c}{c} + \ldots,$$

which is better written

$$F(c) = \{a_0 - f(0)\} + a_1\cos c + a_2\cos 2c + \ldots,$$

where

$$F(c) = c\int_0^{\frac{1}{2}\pi} f'(c\sin\theta)\,d\theta.$$

Treating this as a Fourier series, we derive the coefficients as

$$a_0 - f(0) = \frac{1}{\pi}\int_0^{\pi} F(c)\,dc, \quad a_n = \frac{2}{\pi}\int_0^{\pi} F(c)\cos nc\,dc.$$

The series for $f(x)$ may now be considered as known, at least formally. It is evident that in practice the difficulties will arise from the integrations connected with $F(c)$. As an instance of a soluble case we may consider the parabolic form $f(x) = \frac{1}{2}x^2$. This gives $f'(x) = x$,

$$F(c) = c\int_0^{\frac{1}{2}\pi} c\sin\theta\,d\theta = c^2,$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} c^2\,dc = \frac{\pi^2}{3}, \quad a_n = \frac{2}{\pi}\int_0^{\pi} c^2\cos nc\,dc = (-1)^n\,\frac{4}{n^2},$$

$$\tfrac{1}{2}x^2 = \frac{\pi^2}{3} - 4\{J_0(x) - \tfrac{1}{4}J_0(2x) + \tfrac{1}{9}J_0(3x) - \ldots\},$$

valid in the range

$$0 < x < \pi.$$

CHAPTER X

Allied Functions

10·1. Functions of the third kind.

It was suggested by Nielsen that in honour of Hankel the symbol H should be used to denote the function $J_n \pm iY_n$. We accordingly have the somewhat cumbrous notation

$$(1) \quad H_n^{(1)}(x) = J_n(x) + iY_n(x), \quad H_n^{(2)}(x) = J_n(x) - iY_n(x).$$

These being linear combinations of J_n and Y_n are necessarily solutions of Bessel's equations of order n. They accordingly satisfy exactly the same recurrence formulæ as J_n and Y_n. Moreover, since Y_n was defined by means of J_n, it follows that any one of J_n, Y_n, H_n must be expressible as a linear combination of the other two. There is the further relation connecting any two solutions of the same equation.

The H functions find their chief application in the theory of the subject. The assessment of their importance in practice is probably largely subjective. The standard work by Gray, Mathews and MacRobert does not even mention them; but they occasionally appear in Rayleigh's work, and they are fairly freely used in Carslaw, *Mathematical Theory of the Conduction of Heat.* They are readily calculable from tables of J_n and Y_n, and tables of orders 0, 1, $\frac{1}{3}$ are given in Watson, *Theory of Bessel Functions.*

10·2. Kelvin's functions.

There are a round score or so of functions closely allied to Bessel functions. For the most part their interest is mathematical; but outstanding exceptions are the ber and bei functions and their congeners. They play an important part in alternating current theory; they have also been used in the two-dimensional motion of a viscous liquid.

The equation for $I_0(t)$ is

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} - y = 0.$$

Put $t = x(i)^{\frac{1}{2}}$ and the equation becomes

(1) $$\frac{d^2y}{i\,dx^2} + \frac{1}{ix}\frac{dy}{dx} - y = 0,$$

with the solutions $I_0(x\sqrt{i})$ and $K_0(x\sqrt{i})$. The ber and bei functions are defined as follows. Since

$$I_0(t) = 1 + (\tfrac{1}{2}t)^2 + \frac{(\frac{1}{2}t)^4}{(2!)^2} + \cdots ,$$

we have real and imaginary parts in

$$I_0(x\sqrt{i}) = \left\{1 - \frac{(\frac{1}{2}x)^4}{(2!)^2} + \frac{(\frac{1}{2}x)^8}{(4!)^2} - \cdots\right\}$$

$$+ i\left\{(\tfrac{1}{2}x)^2 - \frac{(\frac{1}{2}x)^6}{(3!)^2} + \frac{(\frac{1}{2}x)^{10}}{(5!)^2} - \cdots\right\}$$

(2) $$= \operatorname{ber} x + i \operatorname{bei} x.$$

The names are convenient abbreviations for Bessel-real and Bessel-imaginary. Both ber x and bei x are purely real for real x, and it is easily shown that both series are absolutely convergent for all values of x. Among the more obvious properties we have

(3) $$\operatorname{ber} 0 = 1, \quad \operatorname{bei} 0 = 0,$$

(4) $$\int_0^x x \operatorname{ber} x\,dx = x \operatorname{bei}' x, \quad \int_0^x x \operatorname{bei} x\,dx = -x \operatorname{ber}' x,$$

the latter pair being established from the series. The graphs of both functions oscillate (fig. 10). The functions ker and kei are similarly defined from

(5) $$K_0(x\sqrt{i}) = \operatorname{ker} x + i \operatorname{kei} x,$$

and there is yet another pair, her and hei, defined from Hankel's function. It can be left to the reader to prove that the four functions ber, bei, ker and kei are the solutions of the fourth order differential equation

$$\{\vartheta^2(\vartheta - 2)^2 + e^{4\theta}\}y = 0.$$

The application of the method of Frobenius then shows that the roots of the indicial equation are $r = 0, 0, 2, 2$. It follows that there is a solution beginning with r^0; this is ber x. The second solution for the zero index certainly contains a logarithmic term; this is ker x. Similarly the second solution for the index 2 certainly contains a logarithmic term; this is kei x. Finally, in accordance with theory, the first solution corresponding to the index 2 may or may not contain a logarithmic term. It happens that it does not, and the solution is bei x. All the functions have been generalized for order and argument. Those which have been mentioned are of zero order, in which case it is customary to omit the suffix denoting this.

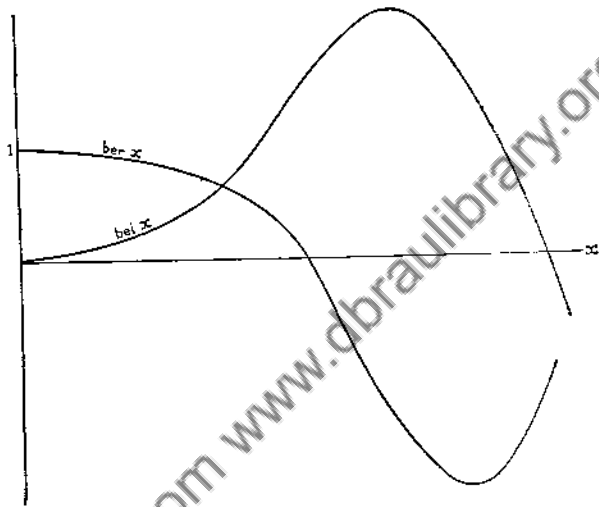

Fig. 10.—March of ber x and bei x

10·3. Current distribution in a conductor.

Probably the simplest application is to the current distribution over the cross-section of a conductor. The argument is based on the two circuital theorems of Ampère and Faraday. Let H be the magnetic intensity at distance r from the axis of a conductor of radius c. The work done in carrying unit pole round the circle of radius r is $W = 2\pi rH$. For a circle of radius $r + dr$ we have $W + dW$. The difference dW tallies with 4π times the current through the annulus. Hence

$$\frac{d}{dr}(2\pi rH)\,dr = 4\pi(2\pi r\,dr)\sigma,$$

where σ is the variable current density. Thus

$$\frac{1}{r}\frac{d}{dr}(rH) = 4\pi\sigma \quad \text{and} \quad \frac{1}{r}\frac{\partial}{\partial r}\left\{r\,\frac{\partial H}{\partial t}\right\} = 4\pi\,\frac{\partial \sigma}{\partial t}.$$

Taking a longitudinal section of unit length, the annulus provides two rectangles measuring 1 by dr. The magnetic induction through one of these is $\mu H dr$. Taking the upper rectangle only, the potential drop along the lower edge is $E = \rho\sigma$, where ρ is the resistivity. For the upper edge we have similarly $E + dE$, and for the line integral round the rectangle we have

$$-\frac{\partial E}{\partial r}\,dr = -\rho\frac{\partial \sigma}{\partial r}\,dr = -\mu\frac{\partial H}{\partial t}\,dr,$$

where μ is the permeability. The elimination of H gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial \sigma}{\partial r}\right\} = \frac{4\pi\mu}{\rho}\frac{\partial \sigma}{\partial t}.$$

Presuming that the alternating current density is given by

$$\sigma = R\exp i(\omega t + \phi), \quad \frac{\partial \sigma}{\partial t} = i\omega\sigma,$$

we have
$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{dR}{dr}\right\} - \frac{4i\pi\mu\omega}{\rho}R = 0,$$

or,
$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - ik^2R = 0, \quad k^2 = \frac{4\pi\mu\omega}{\rho}.$$

The solution is evidently

$$R = C\{\operatorname{ber}(kr) + i\operatorname{bei}(kr)\},$$

which remains finite on the axis, C being some arbitrary constant.

Consideration of the phase of σ shows that the current is not necessarily everywhere in the same direction, since the functions ber and bei may be of the same or opposite signs, according to the value of kr; in fact, if ω and accordingly k is large, the current density vector may revolve more than once.

If σ_c denote the current density at the surface, we have

$$\sigma_c \propto C\{\operatorname{ber}(kc) + i\operatorname{bei}(kc)\},$$

the time periodicity factor being understood. The total current through a circle of radius r is

$$\int_0^r 2\pi\sigma r\,dr = \frac{2\pi C}{k^2}\int_0^r \{kr\operatorname{ber}(kr)\,d(kr) + ikr\operatorname{bei}(kr)\,d(kr)\}$$

$$= \frac{2\pi C}{k^2}\,[kr\{\operatorname{bei}'(kr) - i\operatorname{ber}'(kr)\}].$$

If we denote this by P, and let Q denote the corresponding quantity for the whole cross-section, we have

$$\left|\frac{P}{Q}\right|^2 = \left(\frac{r}{c}\right)^2 \frac{\mathrm{bei}'^2(kr) + \mathrm{ber}'^2(kr)}{\mathrm{bei}'^2(kc) + \mathrm{ber}'^2(kc)}.$$

If σ_m denote the mean current density we have

$$\sigma_m = \frac{Q}{\pi c^2} = \frac{2C}{kc}\{\mathrm{bei}'(kc) - i\,\mathrm{ber}'(kc)\}.$$

It follows that the constant C may be considered known if either σ_c or σ_m is known. It is customary to write

$$\sigma_c \rho = Q(R + i\omega L),$$

and call R the equivalent or effective resistance, L the internal self-inductance, ωL the effective reactance. Whatever they may be called, we have

$$R + i\omega L = \frac{ik\rho}{2\pi c} \frac{\mathrm{ber}(kc) + i\,\mathrm{bei}(kc)}{\mathrm{ber}'(kc) + i\,\mathrm{bei}'(kc)}.$$

The importance of the absolute value of the ratio P/Q lies in its ability to account for the "skin effect", whereby at high frequencies the current tends to desert the centre of the conductor. As no adequate explanation can be given without the use of asymptotic expansions we postpone the matter. The various types of function have been frequently calculated and they are usually published in the British Association Reports.

10·4. Asymptotic expansions.

As these have several times been mentioned in the course of the book it is as well that a word should be said on the matter. It is usually important to know how a function behaves for large values of the argument, especially if tabulation is in view. Tabulation is a tedious business anyway, and a formula that will give an approximate result is invaluable, especially if at the same time it gives an indication of the magnitude of the error. The series expansion of a function may be perfectly valid for all values of the argument, and yet useless for purposes of computation on account of its slow convergence.

The obvious suggestion for a way out of the difficulty is to expand the function in powers of x^{-1}, so that as x gets large the terms get small. The answer is that such an expansion is not necessarily valid. A function that is defined by a differential equation is very much tied in its permissible ways of expression. The reader is probably already familiar with asymptotic formulæ; thus $(1 + n^{-1})^n$ is asymptotic to e. An asymptotic series is something different. In spite of the fact that a valid infinite series in descending powers of the variable possibly does not exist for some given function, it may still be possible by some means to obtain such an expansion formally. The " means " is almost invariably a contour integral using the complex variable $z = x + iy$, and the resulting expansion is divergent. All the same we can always say that the function equals the first n terms plus a remainder, the remainder being defined at worst as the difference between the function and the sum of the first n terms. Everything depends on this remainder, whose value is a function of n and the variable. If it can be shown that, for a fixed n, R tends to zero as the variable increases, then the first n terms constitute the asymptotic expansion of the function, and R gives the magnitude of the error.

An outstandingly good example of an asymptotic series is provided by the error function. An integration by parts gives

$$\int e^{-x^2}dx = \tfrac{1}{2}\int x^{-1}e^{-x^2}d(x^2) = -\frac{e^{-x^2}}{2x} - \int \frac{e^{-x^2}}{2x^2}\,dx.$$

Making this definite we have

$$\int_x^{\infty} e^{-x^2}dx = \frac{e^{-x^2}}{2x} - \int_x^{\infty} \frac{e^{-x^2}}{2x^2}\,dx,$$

and more generally

$$\int_x^{\infty} \frac{e^{-x^2}}{x^{2r}}\,dx = \frac{e^{-x^2}}{2x^{2r+1}} - (2r+1)\int_x^{\infty} \frac{e^{-x^2}}{2x^{2r+2}}\,dx.$$

A repeated application of the last formula gives us

$$\int_x^{\infty} e^{-x^2}dx = \frac{e^{-x^2}}{2x}\left\{1 - \frac{1}{2x^2} + \frac{1\cdot 3}{(2x^2)^2} - \frac{1\,.\,3\,.\,5}{(2x^2)^3} + \,.\,.\,.\right\}.$$

Inspection shows that for any value of x that may be considered large, say 10, the magnitude of the terms decreases with unusual rapidity. Further inspection shows that, no matter how large x may be, the numerators will ultimately overtake the denominators and the terms will then be on their way to increasing beyond all bounds. The series if pursued is divergent; but if we examine the end of the sequence on stopping at the rth term we find that, apart from the factor $\sqrt{\pi}(-)^{r-1}$, we finish up with

$$\frac{e^{-x^2}}{2x}\frac{\Gamma(r-\frac{1}{2})}{x^{2(r-1)}}-\Gamma(r+\tfrac{1}{2})\int_x^\infty \frac{e^{-x^2}\,dx}{x^{2r}}.$$

The integral repays examination. The factor e^{-x^2} steadily decreases throughout the range, showing that the integral must be less than

$$\Gamma(r+\tfrac{1}{2})e^{-x^2}\int_x^\infty x^{-2r}\,dx=\frac{e^{-x^2}}{2x}\frac{\Gamma(r-\frac{1}{2})}{x^{2(r-1)}}.$$

The integral remainder is thus seen to be less than our rth term. The comforting fact emerges that, if we decide to approximate by stopping at the rth term, the error is smaller in magnitude than the last term retained. Putting it into figures, if x is as low as 3 and we decide to retain only two terms, we can bank on the error being below $e^{-9}/108$, which is of the order 10^{-6}. Comparing this with the alternative procedure of expanding e^{-x^2} and integrating termwise, the latter is evidently out of the running. It only remains to add that, if we calculate Erf (x) rather than Erfc (x) we have

$$\int_0^x e^{-x^2}\,dx=\int_0^\infty e^{-x^2}\,dx-\int_x^\infty e^{-x^2}\,dx=\tfrac{1}{2}\sqrt{\pi}-\int_x^\infty e^{-x^2}\,dx.$$

10·5. Solution by definite integrals.

Back in Chapter II we mentioned the matter of solving differential equations by means of definite integrals. We shall now give an elementary exposition of this rather highbrow method by applying it to a convenient form of Bessel's equation and eventually linking up with the asymptotic expansions. We propose to examine our chances of solving

$$\frac{d^2y}{dx^2}+\frac{2n+1}{x}\frac{dy}{dx}+y=0,$$

which is the equation for $x^{-n}C_n(x)$, by means of the integral

$$y=\int Te^{ixt}\,dt.$$

Here T is a function of t alone, and the integral is eventually to be made definite, thus becoming a function of x alone.

Whenever we attempt to solve an equation by using an integral, much naturally depends on choosing a suitable integrand; here of course it is experience that counts. The above integrand may be considered reasonable on the grounds that the cylinder functions behave very much like trigonometrical functions for large values of x.

This substitution poses two problems, both of which are answered in the working. Firstly, what is to be the precise form of T; secondly, what are to be the limits of integration. In the meantime we have by differentiation and partial integration

$$y = \left[T\frac{e^{ixt}}{ix}\right] - \int T'\frac{e^{ixt}}{ix}\,dt,$$

$$\frac{dy}{dx} = \int iTte^{ixt}\,dt,$$

$$\frac{d^2y}{dx^2} = -\int Tt^2e^{ixt}\,dt$$

$$= -\left[Tt^2\frac{e^{ixt}}{ix}\right] + \int(t^2T' + 2tT)\frac{e^{ixt}}{ix}\,dt.$$

Putting these in the equation, we have with a little rearrangement

$$[T(1-t^2)e^{ixt}] - \int\{T'(1-t^2) + (2n-1)Tt\}e^{ixt}\,dt \equiv 0.$$

The first term, in the square bracket, is presumed to be taken at the two limits of integration, which are not yet known. The obvious way of satisfying the identity is to make both the square bracket and the curly bracket vanish separately. To accomplish the latter means solving a small differential equation and we get

$$\frac{T'}{T} = -\frac{(2n-1)t}{1-t^2}, \quad T = (1-t^2)^{n-\frac{1}{2}}.$$

This fixes the form of T and one of the two problems is answered. The square bracket obviously vanishes for $t = \pm 1$; and provided x is a positive real, it tends to zero also as t approaches $1 + i\infty$. This puts us in possession of the two solutions

$$y_1 = \int_{-1}^{1} e^{ixt}(1-t^2)^{n-\frac{1}{2}}\,dt, \quad y_2 = \int_{1}^{1+i\infty} e^{ixt}(1-t^2)^{n-\frac{1}{2}}\,dt.$$

It remains to interpret these.

Concerning the former, it fails to converge at either limit unless $n > -\frac{1}{2}$. Otherwise, if we substitute $t = \sin\theta$ we have

$$2\int_0^{\frac{1}{2}\pi} \cos(x\sin\theta)\cos^{2n}\theta\,d\theta = y_1 = x^{-n}C_n(x).$$

Apart from an irrelevant numerical factor, we have evidently reached $J_n(x)$ as defined by Bessel's second integral.

10·6. The asymptotic solution.

The second solution is more interesting, if for no reason other than its being complex and so providing two solutions by its real and imaginary parts. For positive real x we put

$$t = 1 + \frac{iv}{x}, \quad dt = \frac{idv}{x}, \quad 1 - t^2 = -\frac{iv}{x}\left(2 + \frac{iv}{x}\right),$$

so that the limits for v are 0 and ∞. We then have

$$y_2 = \int_0^\infty \exp(ix - v)\left\{-\frac{iv}{x}\left(2 + \frac{iv}{x}\right)\right\}^{n-\frac{1}{2}} \frac{i\,dv}{x},$$

or,

$$-2y_2 = e^{ix}\left(-\frac{2i}{x}\right)^{n+\frac{1}{2}}\int_0^\infty e^{-v}\left\{v\left(1 + \frac{iv}{2x}\right)\right\}^{n-\frac{1}{2}} dv.$$

Since

$$-i = \text{cis}(-\tfrac{1}{2}\pi) = \exp(-\tfrac{1}{2}i\pi),$$

we can write the multiplier of the integral as

$$\left(\frac{2}{x}\right)^{n+\frac{1}{2}} \exp i\{x - (n + \tfrac{1}{2})\tfrac{1}{2}\pi\}.$$

As for the integrand, we can expand by the binomial and integrate termwise. The interesting fact presents itself, that if n is half an odd integer (a phrase that by this time has a familiar ring about it), the binomial expansion terminates and the result is achieved in finite terms. Testing this out for $n = \frac{1}{2}$, we have

$$-2y_2 = -2x^{-\frac{1}{2}}C_{\frac{1}{2}}(x) = \frac{2}{x}\exp i(x - \tfrac{1}{2}\pi)\int_0^\infty e^{-v}\,dv$$

$$= \frac{2}{x}(\sin x - i\cos x).$$

Apart from irrelevant numerical factors, these evidently give $J_{\frac{1}{2}}$ or $Y_{\frac{1}{2}}$.

If n is not the half of an odd integer, we have the real part of the integral as

$$\int_0^\infty e^{-v}v^{n-\frac{1}{2}}\left\{1-\frac{(n-\frac{1}{2})(n-\frac{3}{2})}{2!(2x)^2}v^2+\ldots\right\}dx$$

$$=\Gamma(n+\tfrac{1}{2})\left\{1-\frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})}{2!(2x)^2}+\ldots\right\}$$

$$=P\Gamma(n+\tfrac{1}{2}).$$

Similarly for the imaginary part we have

$$\int_0^\infty e^{-v}v^{n-\frac{1}{2}}\left\{\frac{(n-\frac{1}{2})}{2x}v-\frac{(n-\frac{1}{2})(n-\frac{3}{2})(n-\frac{5}{2})}{3!(2x)^3}+\ldots\right\}dv$$

$$=\Gamma(n+\tfrac{1}{2})\left\{\frac{n^2-\frac{1}{4}}{2x}-\frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})(n^2-\frac{25}{4})}{3!(2x)^3}+\ldots\right\}$$

$$=Q\Gamma(n+\tfrac{1}{2}).$$

The P and Q series are asymptotic. If pursued they ultimately diverge; but since they were derived from the binomial theorem, they can always be cut short with a remainder. If this was anything more than an elementary exposition we should be under the moral obligation of investigating that remainder; but we will waive the point.

If we multiply $-2y_2$ by $(\frac{1}{2}x)^n/\sqrt{\pi}\Gamma(n+\frac{1}{2})$ and pick out the real and imaginary parts, we have the asymptotic formulæ

(1) $$J_n(x)=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}(P\cos\beta-Q\sin\beta),$$

(2) $$Y_n(x)=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}(P\sin\beta+Q\cos\beta),$$

where

(3) $$\beta=x-\tfrac{1}{2}(n+\tfrac{1}{2})\pi.$$

It is readily verified that these give the usual formulæ exactly when n is half an odd integer.

10·7. The modified functions.

Seeing that Bessel's equation was previously modified by writing ix for x, we have $x^{-n}I_n$ and $x^{-n}K_n$ as solutions of

$$\frac{d^2y}{dx^2}+\frac{2n+1}{x}\frac{dy}{dx}-y=0.$$

Hence also the integral $\qquad y = \int e^{-xt} T\, dt$

is a solution, provided that

$$[T(1-t^2)e^{-xt}] + \int e^{-xt}\{T'(t^2-1) - (2n-1)Tt\}\, dt = 0.$$

The square bracket is zero when $t = 1, -1$ and tends to zero as t tends to infinity. We therefore examine the integral

$$y = \int_1^\infty e^{-xt}(t^2-1)^{n-\frac{1}{2}}\, dt, \quad t = 1 + \frac{v}{x},$$

$$= \int_0^\infty \exp(-x-v)\left\{\frac{v}{x}\left(2+\frac{v}{x}\right)\right\}^{n-\frac{1}{2}} \frac{dv}{x}.$$

$$2y = e^{-x}\left(\frac{2}{x}\right)^{n+\frac{1}{2}} \int_0^\infty e^{-v}\left\{v\left(1+\frac{v}{2x}\right)\right\}^{n-\frac{1}{2}} dv$$

$$= e^{-x}\left(\frac{2}{x}\right)^{n+\frac{1}{2}} \Gamma(n+\tfrac{1}{2})\left\{1 + \frac{4n^2-1^2}{1!(8x)} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!(8x)^2} + \dots\right\}.$$

We know that yx^n must be a linear combination of I_n and K_n; but as I_n tends to infinity with x we conclude that it is not present here. After re-arranging some factors we have the asymptotic formula

$$(1) \qquad K_n(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}\left\{1 + \frac{4n^2-1^2}{1!(8x)} + \dots\right\}.$$

This can be checked for the value $n = \frac{1}{2}$.

The corresponding formula for I_n is

$$(2) \qquad I_n(x) = \frac{e^x}{(2\pi x)^{\frac{1}{2}}}\left\{1 - \frac{4n^2-1^2}{1!(8x)} + \dots\right\}.$$

The asymptotic formulæ for the ber and bei functions of general order are rather fearsome affairs. Even those for zero order are not very alluring. We give them here without commenting further; they can be deduced from the asymptotic formula for $I_0(x)$.

$$\operatorname{ber} x = \frac{e^\beta}{\sqrt{(2\pi x)}}\cos\alpha, \quad \operatorname{bei} x = \frac{e^\beta}{\sqrt{(2\pi x)}}\sin\alpha,$$

$$\alpha = \frac{x}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{8\sqrt{2x}} - \dots,$$

$$\beta = \frac{x}{\sqrt{2}} + \frac{1}{8\sqrt{2x}} - \dots.$$

Adverting to the skin effect mentioned in 10·3 we have by definition

$$I_0(kxi^{\frac{1}{2}}) = \operatorname{ber} kx + i \operatorname{bei} kx.$$

Differentiating with respect to x, we have in absolute values

$$| I_1(kxi^{\frac{1}{2}}) | = | \operatorname{bei}' kx - i \operatorname{ber}' kx |$$

since the absolute value of $i^{\frac{1}{2}}$ is unity. The real part of $i^{\frac{1}{2}}$ is $2^{-\frac{1}{2}}$ and on using the asymptotic value of I_1 we have

$$\left| \frac{P}{Q} \right| = \left(\frac{r}{c}\right)^{\frac{1}{2}} \exp \frac{k}{\sqrt{2}} (r - c).$$

For any fixed values of r and c the exponent is negative, so that with increasing k the absolute value $| P/Q |$ tends to zero, even when r and c are quite close.

BIBLIOGRAPHICAL NOTE

A few words on bibliography and the problems may be of interest. For the most part the problems are obviously the common property of the mathematical world. A few may be original, but I stake no claim to that. For one thing, the claim is scarcely worth making; for another, I should risk being non-suited. They came when bidden, possibly from the recesses of memory, in which case they left their ancestry behind.

Take, for example, the transverse vibrations of a non-uniform rod. Anybody conversant with the literature of the subject knows that it was originally tackled by Kirchhoff. It was done again by J. W. Nicholson and it appeared in *Proc. Roy. Soc.* xciii A (1917). Five years later the same journal published another account by D. Wrinch in Vol. 101 (1922).

Similarly Bernoulli's chain problem was modified by Greenhill, as stated in the text. The matter was investigated experimentally by J. R. Airey who gave his results in *Phil. Mag.* (6) xxi, 1911, p. 736. He adds without proof that the uniform chain with end load is soluble in terms of J and Y of orders 0, 1, 2; and he states correctly that the problem figures in Routh, *Advanced Rigid Dynamics*. It also figures as No. 32 in the miscellaneous examples at the end of Gray's treatise mentioned below.

The lengthening pendulum, which I regard as an outstandingly good illustration, figures in the last-mentioned source as No. 49 and the authors quote Lecornu in the *Comptes Rendus* of 15th January, 1894. This I am pretty sure I first met in a record of a conversation between Einstein and Max Planck, but I have failed to trace it. The upshot is that if anyone feels aggrieved at having their material pressed into my service, I hasten to assure them that it was done inadvertently and I am prepared to make reasonable amends. And now for the books, to all of which I am indebted.

1. Watson: *Theory of Bessel Functions.*

This impeccable and exhaustive treatise is the standard text for mathematicians in English. It contains ninety pages of tables, a long bibliography, and not a graph in all its eight hundred odd pages.

2. Gray, Mathews and MacRobert: *A Treatise on Bessel Functions and their Applications to Physics.*

This well-known work contains fifty pages of tables, a bibliography, a graph and a large collection of miscellaneous examples. The subject is treated by contour integrals with more rigour than herein; the applications take up half the book and are more advanced than mine.

3. Riemann: *Partielle Differentialgleichungen.*

One of the world's most delightful mathematical books; sometimes known as Riemann-Weber. To those familiar with the older edition, the latest issue looks like vandalism.

4. E. B. Wilson: *Advanced Calculus.*

This miniature encyclopædia is invaluable to those wishing to improve on the calculus of their student years. I am indebted to the author for stabilizing my belief that the recurrence formulæ made a suitable approach and that solution by integrals, leading to the asymptotic formulæ, was feasible by elementary means.

5. Jahnke und Emde: *Funktionentafeln mit Formeln und Kurven.*

The reputation of this book has grown steadily and deservedly over the last thirty years. It is now procurable printed in English and German on opposite pages. It contains an unparalleled amount of information on tabulated functions, together with excellent graphs. It should be in the possession of all who meditate applications of higher mathematics.

6. Prescott: *Applied Elasticity.*

This gives the necessary analysis and the discussion of such of our problems as depend on the theory of elasticity. Its form is more easily assimilable than the standard work by Love. It also treats the instability of the deep girder.

7. Case: *Strength of Materials.*

This is useful for the ordinary theory of beams and struts, and it gives the theory of the instability of the deep cantilever.

8. Carslaw: *Conduction of Heat.*

The original work was later issued in two volumes, the first dealing with Fourier series and the second with heat conduction. The latter gives the applications of Bessel functions.

9. Lamb: *Hydrodynamics.*

This standard work gives several applications of Bessel functions to fluid motion.

10. Rayleigh: *Theory of Sound.*

Most of the vibration problems will be found treated at greater length there.

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